From Sturm-Liouville problems to fractional and anomalous diffusions

Mirko D'Ovidio*
Sapienza University of Rome

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Abstract Some fractional and anomalous diffusions are driven by equations involving fractional derivatives in both time and space. Such diffusions are processes with randomly varying times. In representing the solutions to those equations, the explicit laws of certain stable processes turn out to be fundamental. This paper directs one's efforts towards the explicit representation of solutions to fractional and anomalous diffusions related to Sturm-Liouville problems of fractional order associated to fractional power function spaces. Furthermore, we study a new version of the Bochner's subordination rule and we establish some connections between subordination and space-fractional operator.

Keywords: Anomalous diffusion, Sturm-Liouville problem, Stable subordinator, Mellin convolution, Fox functions.

1 Introduction and main results

In recent years, many researchers have shown their interest in fractional and anomalous diffusions. The term fractional is achieved by replacing standard derivatives w.r.t. time t with fractional derivatives, for instance, those of Riemann-Liouville or Dzhrbashyan-Caputo. Anomalous diffusion occurs, according to most of the significant literature, when the mean square displacement (or time-dependent variance) is stretched by some index, say $\alpha \neq 1$ or, in other words proportional to a power α of time, for instance t^α. Such anomalous feature can be found in transport phenomena in complex systems, e.g. in random fractal structures (see Giona and Roman [22]).

Fractional diffusions have been studied by several authors. Wyss [60], Schneider and Wyss [56] and later Hilfer [26] studied the solutions to the heat-type fractional diffusion equation and presented such solutions in terms of Fox's functions. For the same equation, up to some scaling constant, Beghin and Orsingher [6]; Orsingher and Beghin [49] represented the solutions by means of stable densities and found the explicit representations only in some cases. Different boundary value problems have also been studied by Metzler and Klafter [45]; Beghin and Orsingher [5]. In the papers by Mainardi et al. [36, 38, 39] the authors presented the solutions to space-time fractional equations by means of Wright functions or Mellin-Barnes integral

*Dipartimento di Scienze di Base e Applicate per l'Ingegneria, via A. Scarpa 10, 00161 Rome (Italy), e-mail:mirko.dovidio@uniroma1.it

representations, that is Fox's functions. See also Mainardi et al. [40] for a review on the Mainardi-Wright function.

For a general operator $\mathscr{A}$ acting on space, several results can also be listed. Nigmatullin [48] gave a physical interpretation when $\mathscr{A}$ is the generator of a Markov process whereas Kochubei [30, 31] first introduced a mathematical approach. Zaslavsky [61] introduced the fractional kinetic equation for Hamiltonian chaos. Baeumer and Meerschaert [1] studied the problem when $\mathscr{A}$ is an infinitely divisible generator on a finite dimensional space. For a short survey of these results see Nane [47].

In general, the stochastic solutions to fractional diffusion equations can be realized through subordination. Indeed, for a guiding process $X(t)$ with generator $\mathscr{A}$ we have that $X(V(t))$ is governed by $\partial_t^\beta u = \mathscr{A} u$ where the process $V(t)$, $t>0$ is an inverse or hitting time process to a β-stable subordinator (see Baeumer and Meerschaert [1]). The time-fractional derivative comes from the fact that $X(V(t))$ can be viewed as a scaling limit of continuous time random walk where the iid jumps are separated by iid power law waiting times (see Meerschaert and Scheffler [43]; Metzler and Klafter [46]; Roman and Alemany [53]). Results on the subordination principle for fractional evolution equations can also be found in Bazhlekova [4]; Bochner [11].

Anomalous diffusions can also be carried out by considering a fractional operator acting on the space. The problem of finding a suitable representation for a fractional power of a given operator A defined in a Banach space X has a long history. The first definitions of fractional power of the Laplace operator were introduced by Bochner [11]; Feller [20]. For a closed linear operator A, the fractional operator $(-A)^\alpha$ has been investigated by many researchers, see e.g. Balakrishnan [3]; Hövel and Westphal [27]; Komatsu [32]; Krasnosel'skii and Sobolevskii [33]; Watanabe [59]. Although the methods presented differ, each of those papers was primarily based on the integral representation

$$-(-A)^\alpha f = \frac{\sin \pi\alpha}{\pi} \int_0^\infty \lambda^{\alpha-1} (\lambda I - A)^{-1} A\, f\, d\lambda$$

for a well defined f and $0 < \Re\{\alpha\} < 1$ under

$$\begin{array}{ll} (i) & \lambda \in \rho(A) \text{ (the resolvent set of } A\text{) for all } \lambda > 0; \\ (ii) & \|\lambda(\lambda I - A)^{-1}\| < M < \infty \text{ for all } \lambda > 0. \end{array} \tag{1.1}$$

Different definitions can be also given by means of hyper singular integrals, see e.g. Samko [54].

In both cases, time and space fractional equations, the explicit representations of the law of stable processes and, those of the corresponding inverse processes, are fundamental in finding explicit solutions to fractional problems.

In this paper we study time and space fractional problems involving the operator $\mathcal{G}^*$ (see formula (3.1)) which is the adjoint of an infinitesimal generator of non-negative diffusions $\mathcal{G}$. In particular, for $\mathfrak{w}(x) = x^{\gamma\mu-1}$, the second order differential operator

$$\mathcal{G} = \frac{1}{\gamma^2\, \mathfrak{w}(x)} \frac{\partial}{\partial x} x^{\gamma\mu-\gamma+1} \frac{\partial}{\partial x}, \quad \gamma = \pm 1,\ \mu > 0$$

is the operator governing two related diffusions, the squared Bessel process $G_\mu(t)$, $t>0$ (for $\gamma=+1$) and its inverse process $E_\mu(t)$, $t>0$ (also known as reciprocal process of G_μ, for $\gamma=-1$). Due to the fact that $P\{E_\mu(t) < x\} = P\{G_\mu(x) > t\}$ we refer to E_μ as the inverse of G_μ.

For such processes we study the governing equations where the time derivative is replaced by its fractional counterpart and find solutions in bounded and unbounded

domains. For the time-fractional equations on bounded domains we study Sturm-Liouville problems of fractional order associated with fractional power function spaces. A complete orthogonal set of eigenfunctions (w.r.t. the weight function $\mathfrak{w}(x)$) arises naturally as solutions of the second-order differential equations involving $\mathcal{G}$ under appropriate boundary conditions and therefore, we obtain Sturm-Liouville boundary-value problems associated to the killed semigroups of G_μ and E_μ.

The fractional power of $\mathcal{G}^*$ (for $\gamma = +1$, that is the governing operator of the squared Bessel process G_μ) is also examined. We find the explicit representation of $-(-\mathcal{G}^*)^\nu$ for $\nu \in (0,1)$ in terms of Riemann-Liouville derivatives and we discuss the properties of the corresponding subordinated process. Thus, we obtain a representation of the power ν of the composition of non commuting operators (formula (3.1) below).

All fractional problems investigated in this work have stochastic solutions with randomly varying times which are subordinators. Such subordinators are denoted by $\mathfrak{H}^\nu_t$ and $\mathfrak{L}^\nu_t$. From the fact that $P\{\mathfrak{L}^\nu_t < x\} = P\{\mathfrak{H}^\nu_x > t\}$ we say that $\mathfrak{L}^\nu_t$ is the inverse of $\mathfrak{H}^\nu_t$ which is a positively skewed stable process with non-negative increments and therefore non-decreasing paths. This means that $\mathfrak{L}^\nu_t = \inf\{x \geq 0 \, : \, \mathfrak{H}^\nu_x \notin (0,t)\}$ can be regarded as hitting time. We find that, for $\nu = 1/n$, $n \in \mathbb{N}$,

$$\mathfrak{H}^\nu_t \stackrel{law}{=} E_{\mu_1}(E_{\mu_2}(\ldots E_{\mu_n}((\nu t)^{1/\nu})\ldots)), \quad t > 0 \tag{1.2}$$

and

$$\mathfrak{L}^\nu_t \stackrel{law}{=} \left[G_{\mu_1}(G_{\mu_2}(\ldots G_{\mu_n}(t^\nu/\nu)\ldots))\right]^\nu, \quad t > 0 \tag{1.3}$$

for suitable choices of $\boldsymbol{\mu} = (\mu_1, \ldots, \mu_n)$. Furthermore, we show that the compositions (1.2) and (1.3) hold for all $\boldsymbol{\mu} \in \mathscr{P}^n_{n+1}(n!)$ where

$$\mathscr{P}^n_\kappa(\varrho) = \left\{ \bar{\wp} \in \mathbb{R}^n_+ \, : \, \bar{\wp} = \frac{\bar{\upsilon}}{\kappa}, \, \bar{\upsilon} = (\upsilon_1, \ldots, \upsilon_n) \in \mathbb{N}^n, \, \prod_{j=1}^n \upsilon_j = \varrho \right\}$$

with $m, \kappa, \varrho \in \mathbb{N}$. This result permit us to explicitly write the laws of $\mathfrak{H}^{1/n}_t$ and $\mathfrak{L}^{1/n}_t$ and therefore the laws of the processes subordinated by them. In particular, we obtain useful representations of solutions to fractional equations involving general operators but representing anomalous diffusions realized through subordination.

The main results of this work are collected in Section 3. We present auxiliary results and proofs in the remaining sections of the paper.

2 Introductory remarks and notations

We first introduce the following notation:

- s_ν is the law of the symmetric stable process S^ν_t,
- h_ν is the law of the stable subordinator $\mathfrak{H}^\nu_t$,
- l_ν is the law of $\mathfrak{L}^\nu_t$ which is the inverse to $\mathfrak{H}^\nu_t$,
- $g_\mu = g^1_\mu$ is the law of the squared Bessel process starting from zero $G_\mu(t)$,
- $e_\mu = g^{-1}_\mu$ is the law of $E_\mu(t)$ which is the (reciprocal of) inverse to $G_\mu(t)$,
- $H^{m,n}_{p,q}$ is the Fox's function,

- W_+^α and W_-^α are the left and right Weyl's derivatives,
- $\frac{\partial^\alpha}{\partial x^\alpha}$ and $\frac{\partial^\alpha}{\partial(-x)^\alpha}$ are the left and right Riemann-Liouville derivatives,
- ${}^C D_t^\alpha$ is the Dzhrbashyan-Caputo fractional derivative.

We now introduce fractional derivatives and recall their connection with stable densities. The α-stable process $S_t^{\alpha,\theta}$, $t>0$, with law $s_\nu^\theta = s_\nu^\theta(x,t)$, $x\in\mathbb{R}$, $t>0$ has characteristic function

$$E\exp\left(i\,\beta\,S_t^{\alpha,\theta}\right) = \exp\left[-t|\xi|^\alpha\left[1-i\theta\frac{\xi}{|\xi|}\tan\left(\frac{\alpha\pi}{2}\right)\right]\right], \quad \xi\in\mathbb{R} \tag{2.1}$$

with $\alpha\in(0,1)\cup(1,2]$ and $\theta\in[-1,1]$ (see Zolotarev [63]). If $\theta=0$, then we have a symmetric process with $E\exp(i\beta S_t^\alpha) = \exp(-t|\beta|^\alpha)$, $\alpha\in(0,2]$. For the sake of simplicity we will write S_t^α instead of $S_t^{\alpha,0}$. Moreover, we will refer to $\mathfrak{H}_t^\nu = S_t^{\nu,1}$, $t>0$ as the totally (positively) skewed process which is also named stable subordinator. For $n-1<\alpha<n$, $n\in\mathbb{N}$, according to Kilbas et al. [29]; Samko et al. [55], we define

$$(W_-^\alpha f)(x) = \frac{(-1)^n}{\Gamma(n-\alpha)}\frac{d^n}{dx^n}\int_x^\infty (s-x)^{n-\alpha-1}f(s)\,ds, \quad x\in\mathbb{R} \tag{2.2}$$

and

$$(W_+^\alpha f)(x) = \frac{1}{\Gamma(n-\alpha)}\frac{d^n}{dx^n}\int_{-\infty}^x (x-s)^{n-\alpha-1}f(s)\,ds, \quad x\in\mathbb{R} \tag{2.3}$$

which are the right and left Weyl's derivatives by means of which we write the governing equation of $S_t^{\alpha,\theta}$, $t>0$, given by

$$\frac{\partial s_\alpha}{\partial t}(x,t) = {}_\theta D_{|x|}^\alpha s_\alpha(x,t), \quad x\in\mathbb{R},\ t>0 \tag{2.4}$$

where

$${}_\theta D_{|x|}^\alpha = \frac{1}{2\cos\alpha\pi/2}\left[\kappa\,W_-^\alpha + (1-\kappa)\,W_+^\alpha\right] \tag{2.5}$$

and $0\le\kappa=\kappa(\theta)\le 1$ (see e.g. Benson et al. [7]; Chaves [13]; Mainardi et al. [36]; Orsingher and Zhao [50]). For $\theta=0$ (that is $\kappa=\kappa(0)=1/2$) we obtain the Riesz derivative

$${}_0 D_{|x|}^\alpha = \frac{\partial^\alpha}{\partial|x|^\alpha} \tag{2.6}$$

which is the governing operator of the symmetric process S_t^α, $t>0$. The Riemann-Liouville derivatives

$$\frac{d^\alpha f}{d(-x)^\alpha}(x) = (W_-^\alpha f)(x), \quad x\in\mathbb{R}_+ \tag{2.7}$$

and

$$\frac{d^\alpha f}{dx^\alpha} = \frac{1}{\Gamma(n-\alpha)}\frac{d^n}{dx^n}\int_0^x (x-s)^{n-\alpha-1}f(s)\,ds, \quad x\in\mathbb{R}_+ \tag{2.8}$$

are defined by restricting the Weyl's derivatives (2.2) and (2.3) to $(0,+\infty)$. For $\alpha\in\mathbb{N}$, the fractional derivatives above become ordinary derivatives and

$$\frac{d^\alpha}{dx^\alpha} = (-1)^\alpha\frac{d^\alpha}{d(-x)^\alpha}. \tag{2.9}$$

We also introduce the Dzhrbashyan-Caputo fractional derivative

$$ {}^cD_t^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-s)^{n-\alpha-1} \frac{d^n f}{ds^n}(s)\, ds \tag{2.10}$$

defined for $n-1 < \alpha < n$, $n \in \mathbb{N}$, which is related to (2.8) as follows (see Gorenflo and Mainardi [24] and Kilbas et al. [29])

$$ {}^cD_t^\alpha f(t) = \frac{d^\alpha f}{dt^\alpha}(t) - \sum_{k=0}^{n-1} \frac{d^k f}{dt^k}(t)\bigg|_{t=0^+} \frac{t^{k-\alpha}}{\Gamma(k-\alpha+1)}. \tag{2.11}$$

From the relation

$$Pr\{\mathfrak{L}_t^\nu < x\} = Pr\{\mathfrak{H}_x^\nu > t\}, \tag{2.12}$$

according to [1; 16; 42; 44], we define the inverse process $\mathfrak{L}_t^\nu$, $t>0$ with law $l_\nu = l_\nu(x,t)$, $x,t>0$. As already mentioned before, $\mathfrak{H}_t^\nu$, $t>0$ is the ν-stable subordinator, $\nu \in (0,1)$ with law, say, $h_\nu = h_\nu(x,t)$, $x,t>0$. The process $\mathfrak{H}_t^\nu$, $t>0$ is a process with non-negative, independent and homogeneous increments (see Bertoin [8]) whereas, the inverse process $\mathfrak{L}_t^\nu$, $t>0$ has non-negative, non-stationary and non-independent increments (see Meerschaert and Scheffler [43]). Stable subordinators and their inverse processes are characterized by the following Laplace transforms:

$$E\exp(-\lambda \mathfrak{H}_t^\nu) = \exp(-t\lambda^\nu), \qquad E\exp(-\lambda\mathfrak{L}_t^\nu) = E_\nu(-\lambda t^\nu) \tag{2.13}$$

and

$$\mathcal{L}[h_\nu(x,\cdot)](\lambda) = x^{\nu-1}E_{\nu,\nu}(-\lambda x^\nu), \qquad \mathcal{L}[l_\nu(x,\cdot)](\lambda) = \lambda^{\nu-1}\exp(-x\lambda^\nu) \tag{2.14}$$

where the entire function

$$E_{\alpha,\beta}(z) = \sum_{k\geq 0} \frac{z^k}{\Gamma(\alpha k+\beta)}, \quad z\in\mathbb{C},\ \Re\{\alpha\}>0,\ \beta\in\mathbb{C} \tag{2.15}$$

is the generalized Mittag-Leffler function for which

$$\int_0^\infty e^{-\lambda z} z^{\beta-1} E_{\alpha,\beta}(-\mathfrak{c}z^\alpha)\, dz = \frac{\lambda^{\alpha-\beta}}{\lambda^\alpha+\mathfrak{c}}, \quad \Re\{\lambda\} > |\mathfrak{c}|^{1/\alpha},\ \Re\{\mathfrak{c}\}>0$$

and $E_\alpha(z) = E_{\alpha,1}(z)$ is the Mittag-Leffler function. From (2.13) we immediately verify that the law h_ν satisfies the fractional equation $-\frac{\partial}{\partial t}h_\nu(x,t) = \frac{\partial^\nu}{\partial x^\nu}h_\nu(x,t)$ whereas, for the law of $\mathfrak{L}_t^\nu$, from (2.14) we have that $\frac{\partial^\nu}{\partial t^\nu}l_\nu(x,t) = -\frac{\partial}{\partial x}l_\nu(x,t)$. Such density laws can not be represented in closed form. In this paper we write

$$h_\nu(x,t) = \frac{1}{\nu t^{1/\nu}} H_{1,1}^{0,1}\left[\frac{x}{t^{1/\nu}}\left| \begin{matrix} (1-\frac{1}{\nu},\frac{1}{\nu}) \\ (0,1) \end{matrix}\right.\right], \quad l_\nu(x,t) = \frac{1}{t^\nu} H_{1,1}^{1,0}\left[\frac{x}{t^\nu}\left| \begin{matrix} (1-\nu,\nu) \\ (0,1) \end{matrix}\right.\right] \tag{2.16}$$

for $x,t>0$, $\nu\in(0,1)$ in terms of H-functions as can be obtained by considering (A.12) and the Mellin transforms (see [16; 36; 37])

$$\mathcal{M}[h_\nu(\cdot,t)](\eta) = \Gamma\left(\frac{1-\eta}{\nu}\right) \frac{t^{\frac{\eta-1}{\nu}}}{\nu\,\Gamma(1-\eta)}, \quad \mathcal{M}[l_\nu(\cdot,t)](\eta) = \frac{\Gamma(\eta)\, t^{\nu(\eta-1)}}{\Gamma(\eta\nu-\nu+1)}. \tag{2.17}$$

Further representations of h_ν and l_ν are given in terms of the Wright function

$$W_{\alpha,\beta}(z) = \sum_{k\geq 0}\frac{z^k}{k!\,\Gamma(\alpha k+\beta)}, \quad z\in\mathbb{C},\ \Re\{\alpha\}>-1,\ \beta\in\mathbb{C}$$

by considering that

$$l_\nu(x,t) = \frac{1}{t^\nu} W_{-\nu,1-\nu}\left(-\frac{x}{t^\nu}\right)$$

and (see D'Ovidio [17]) $xh_\nu(x,t) = tl_\nu(t,x)$.

3 Fractional and Anomalous diffusions

Standard diffusion has the mean squared displacement (or time-dependent variance) which is linear in time. Anomalous diffusion is usually met in disordered or fractal media (see e.g. Giona and Roman [22]) and represents a phenomenon for which the mean squared displacement is no longer linear but proportional to a power α of time with $\alpha \neq 1$. Thus we have superdiffusion ($\alpha > 1$) or subdiffusion ($\alpha < 1$) in which diffusion occurs faster or slower than normal diffusion (see e.g. Uchaǐkin [58]).

3.1 Fractional evolution equations

We begin our analysis by studying anomalous diffusions on $\Omega_a = (0,a)$, $a > 0$, whose governing equations involve the operator

$$\mathcal{G}^* = \frac{1}{\gamma^2}\frac{\partial}{\partial x} x^{\gamma\mu-\gamma+1}\frac{\partial}{\partial x}\frac{1}{\mathfrak{w}(x)} \tag{3.1}$$

($\mathfrak{w}(x) = x^{\gamma\mu-1}$ will play the role of weight function further on) for $\gamma = \pm 1$ and $\mu > 0$.

The solutions to the ordinary problem

$$\frac{\partial u}{\partial t} = \mathcal{G}^* \, u$$

subject to the initial data $u_0 = f$ can be written as $u(x,t) = T_1(t) f(x)$ where $T_1(t) = \exp -t\mathcal{G}^*$. For the subordination principle (see for example Bazhlekova [4]; Bochner [11]) we can write the solutions to the fractional problem

$$\frac{\partial^\nu u}{\partial t^\nu} = \mathcal{G}^* \, u, \quad \nu \in (0,1]$$

($\frac{\partial^\nu u}{\partial t^\nu}$ is defined in (2.8)) subject to the initial condition $u_0 = f$ by considering the convolution operator

$$T_\nu(t) = \frac{1}{t^\nu}\int_0^\infty ds \, H^{1,0}_{1,1}\left[\frac{s}{t^\nu}\middle|\begin{array}{c}(1-\nu,\nu)\\(0,1)\end{array}\right] e^{-s\mathcal{G}^*}$$

where the Fox's function $H^{1,0}_{1,1}$ is the law of the inverse process $\mathfrak{L}^\nu_t$ introduced in the previous section. In order to explicitly write the convolution $T_\nu(t) f(x)$, for $\gamma \neq 0$, $\mu > 0$, we introduce the functions

$$g^\gamma_\mu(x,t) = \mathrm{sign}(\gamma)\frac{1}{t}Q^\gamma_\mu\left(\frac{x}{t}\right) \quad \text{and} \quad \tilde{g}^\gamma_\mu(x,t) = g^\gamma_\mu(x, t^{1/\gamma}) \tag{3.2}$$

where

$$Q^\gamma_\mu(z) = \gamma\frac{z^{\gamma\mu-1}}{\Gamma(\mu)}e^{-z^\gamma}, \quad z > 0, \, \gamma > 0, \, \mu > 0$$

is the well known generalized gamma density or Weibull distribution if $\gamma \in \mathbb{N}$.

Theorem 1. *For $\gamma \neq 0$, $\mu > 0$, $\nu \in (0,1]$, the solution to the fractional p.d.e.*

$$\begin{cases} \frac{\partial^\nu \tilde{u}_\nu^{\gamma,\mu}}{\partial t^\nu} = \mathcal{G}^* \, \tilde{u}_\nu^{\gamma,\mu}, \quad x \in \Omega_\infty, \, t > 0 \\ \tilde{u}_\nu^{\gamma,\mu}(x,0) = \delta(x) \end{cases} \tag{3.3}$$

is given by

$$\tilde{u}_\nu^{\gamma,\mu}(x,t) = \int_0^\infty \tilde{g}_\mu^\gamma(x,s) \, l_\nu(s,t) \, ds \tag{3.4}$$

where $\tilde{g}_\mu^\gamma$ and l_ν are defined in (3.2) and (2.16) respectively.

From the fact that

$$\lim_{\nu \to 1} \mathfrak{L}_t^\nu \overset{a.s}{=} t$$

where t is the elementary subordinator (see Bertoin [8]) we can write

$$\lim_{\nu \to 1} l_\nu(s,t) = \delta(s-t)$$

and thus $\tilde{u}_1^{\gamma,\mu} = \tilde{g}_\mu^\gamma$ are the solutions to (3.3) for $\nu = 1$ and $\gamma \neq 0$, $\mu > 0$. In this case the time-fractional derivative becomes the ordinary derivative $\partial/\partial t$. For $\gamma = \pm 1$ and $\mu > 0$, we obtain that

$$\begin{aligned} \tilde{g}_\mu^1(x,t) &= E^x \delta(G_\mu(t)), \\ \tilde{g}_\mu^{-1}(x,t) &= E^x \delta(E_\mu(t)), \\ \tilde{u}_\nu^{1,\mu}(x,t) &= E^x \delta(G_\mu(L_t^\nu)), \\ \tilde{u}_\nu^{-1,\mu}(x,t) &= E^x \delta(E_\mu(L_t^\nu)) \end{aligned}$$

where $E^x \delta(X_t) = \delta * f_{X_t}(x)$. The process G_μ is a non-negative diffusion satisfying the stochastic differential equation

$$dG_\mu(t) = \mu \, dt + 2\sqrt{G_\mu(t)} \, dB_1(t) \tag{3.5}$$

where $B_1(t)$, $t > 0$ is a Brownian motion with variance $t/2$. The reciprocal gamma law $e_\mu = g_\mu^{-1}$, $\mu > 0$, represents the 1-dimensional marginal law of the process satisfying the stochastic equation

$$dE_\mu(t) = -\left(E_\mu(t) - \frac{1}{\mu - 1} \right) dt + \sqrt{\frac{2\,|E_\mu(t)|^2}{\mu - 1}} \, dB_2(t) \tag{3.6}$$

where $B_2(t)$, $t > 0$ is a standard Brownian motion (see e.g. Bibby et al. [9], Peškir [51]). Due to the global Lipschitz condition on both coefficients, the stochastic equation (3.6) has a unique solution which is a strong Markov process. The process E_μ also appears by considering the integral of a geometric Brownian motion with drift μ, that is

$$\frac{1}{2} E_\mu \overset{law}{=} \int_0^\infty \exp\left(2B(s) - 2\mu s\right) ds \tag{3.7}$$

see Dufresne [19]; Pollack and Siegmund [52]. For $\gamma = 2$, the operator $\mathcal{G}^*$ becomes the governing operator of a 2μ-dimensional Bessel process $R_{2\mu} = G_{2\mu}^{1/2}$.

For the processes G_μ and E_μ introduced above there exist a couple of interesting properties. In particular, we have that

$$Pr\{G_\mu(x) > t\} = Pr\{E_\mu(t) < x\}$$

and therefore we refer to E_μ as the inverse to G_μ, whereas from the fact that

$$E_\mu(t) \overset{law}{=} 1/G_\mu(t)$$

the process E_μ will be also termed reciprocal process of G_μ.

Remark 1. We observe that $\mathfrak{H}_t^\nu$ has non-negative increments and therefore, from (2.12), the inverse process $\mathfrak{L}_t^\nu$ can be regarded as an Hitting time. This does not hold for E_μ being G_μ a diffusion driven by (3.5).

The operator $\mathcal{G}^*$ is the adjoint of the infinitesimal generator

$$\mathcal{G} = \frac{1}{\gamma^2\, \mathfrak{w}(x)} \frac{\partial}{\partial x} x^{\gamma\mu-\gamma+1} \frac{\partial}{\partial x}, \quad \gamma = \pm 1,\, \mu > 0 \tag{3.8}$$

which is a second order differential operator driving the squared Bessel process G_μ if $\gamma = +1$ and the inverse process E_μ if $\gamma = -1$. The Sturm-Liouville eigenvalue problem (see Bochner [10]; Courant and Hilbert [14]) associated with (3.8) leads to the differential equation (see Lemma 2 below)

$$\mathcal{G}\, \bar{\psi}_{\kappa_i} = -(\kappa_i/2)^2\, \bar{\psi}_{\kappa_i}. \tag{3.9}$$

The eigenfunctions

$$\bar{\psi}_{\kappa_i}(x) = x^{\frac{\gamma}{2}(1-\mu)} J_{\mu-1}\left(\kappa_i\, x^{\gamma/2}\right) \tag{3.10}$$

corresponding to different eigenvalues are orthogonal with respect to the weight function $\mathfrak{w}(x) = x^{\gamma\mu-1}$ in the sense that

$$\int_{\Omega_a} \bar{\psi}_{\kappa_i}\left(\frac{x}{a}\right) \bar{\psi}_{\kappa_j}\left(\frac{x}{a}\right) \mathfrak{w}(x) dx = 0, \quad \text{if } \; i \neq j \tag{3.11}$$

on the domain $\Omega_a = (0, a)$, $a > 0$. The eigenvalues are written in terms of $\{\kappa_i\}_{i\in\mathbb{N}}$ which are the zeros of the Bessel function of the first kind (see [35, p. 102])

$$J_\alpha(z) = \sum_{k\geq 0} \frac{(-1)^k\, (z/2)^{\alpha+2k}}{k!\, \Gamma(\alpha+k+1)}, \quad |z| < \infty, \quad |\arg z| < \pi. \tag{3.12}$$

The set of eigenfunctions $\{\bar{\psi}_{\kappa_i}\}_{i\in\mathbb{N}}$ is complete and therefore a piecewise smooth function can be represented by a generalized Fourier series expansion of (3.10). In particular, on the finite domain $\Omega_1 = (0, 1)$, we study the solution to

$$\begin{cases} {}^cD_t^\nu m_\nu^{\gamma,\mu} = \mathcal{G}^*\, m_\nu^{\gamma,\mu}, & x \in \Omega_1,\, t > 0, \\ m_\nu^{\gamma,\mu}(x, 0) = m_0(x), & m_0 \in C(\Omega_1) \\ m_\nu^{\gamma,\mu}(x, t) = 0, & x \in \partial\Omega_1,\; t > 0, \end{cases} \tag{3.13}$$

with $\nu \in (0, 1]$, $\gamma \neq 0$, $\mu > 0$ and arrive at the next result.

Theorem 2. *The solution to the problem* (3.13) *can be written as follows*

$$m_\nu^{\gamma,\mu}(x, t) = \mathfrak{w}(x) \sum_{n=1}^{\infty} c_n\, E_\nu\left(-(\kappa_n/2)^2\, t^\nu\right) \frac{\bar{\psi}_{\kappa_n}(x)}{\|\bar{\psi}_{\kappa_n}\|_{\mathfrak{w}}^2} \tag{3.14}$$

where $\mathfrak{w}(x) = x^{\gamma\mu-1}$ *is the weight function,*

$$c_n = \int_{\Omega_1} m_0(x) \bar{\psi}_{\kappa_n}(x)\, dx, \quad n = 1, 2, \dots \tag{3.15}$$

and $E_\nu(z) = E_{\nu,1}(z)$ *is the Mittag-Leffler function* (2.15).

This result can be easily extended to the case in which the bounded domain is $\Omega_a = (0, a)$ for $a > 0$ and we can write

$$m_\nu^{1,\mu}(x,t) = E^x\, m_0(G_\mu(\mathfrak{L}_t^\nu))\, \mathbf{1}_{(\mathfrak{L}_t^\nu < T_{\Omega_a}(G_\mu))} \tag{3.16}$$

and

$$m_\nu^{-1,\mu}(x,t) = E^x\, m_0(E_\mu(\mathfrak{L}_t^\nu))\, \mathbf{1}_{(\mathfrak{L}_t^\nu < T_{\Omega_a}(E_\mu))} \tag{3.17}$$

where $T_D(X) = \inf\{t \geq 0 :\, X_t \notin D\}$ is an exit time and $E^x \phi(X_t) = \phi * f_{X_t}(x)$. We observe that $\mathbf{1}_{(L_t^\nu < T_{\Omega_a}(X))} = \mathbf{1}_{(t < T_{\Omega_a}(X(L^\nu)))}$. For $a \to \infty$ we obtain the solutions to the problem (3.3). Indeed, we have that $\mathbf{1}_{(L_t^\nu < T_{\Omega_\infty}(G_\mu))} \equiv 1$ and $\mathbf{1}_{(L_t^\nu < T_{\Omega_\infty}(E_\mu))} \equiv 1$. Furthermore, formula (3.11) can be rewritten for $a \to \infty$ as

$$\int_0^\infty \bar{\psi}_{\kappa_i}(x) \bar{\psi}_{\kappa_j}(x) \mathfrak{w}(x) dx = \delta(\kappa_i - \kappa_j)/\kappa_j$$

which leads to the Hankel transforms

$$(\mathcal{H}\, f)(\rho) = \int_0^\infty x J_\nu(\rho x) f(x) dx \quad \text{and} \quad f(x) = \int_0^\infty \rho J_\nu(\rho x) (\mathcal{H}\, f)(\rho) d\rho$$

of a well-defined function f.

3.2 Fractional powers of operators

From the Cauchy integral

$$\frac{1}{2\pi i} \int_\Gamma \frac{f(z)\, dz}{(z - z_0)}$$

we can define an algebraic isomorphism such that a function of a linear bounded operator A is defined as

$$f(A) = \frac{1}{2\pi i} \int_\Gamma f(z)\, R(z, A)\, dz$$

where $R(z, A) = (zI - A)^{-1}$ is the resolvent operator (under conditions (1.1)). In general, for a closed linear operator A in a Banach space X, the definition of A^α for a complex α could be given by means of the Dunford integral

$$A^\alpha = \frac{1}{2\pi i} \int_\Gamma \zeta^\alpha\, (\zeta - A)^{-1}\, d\zeta = -\frac{\sin \pi\alpha}{\pi} \int_0^\infty \lambda^\alpha\, (\lambda + A)^{-1}\, d\lambda$$

where Γ encircles the spectrum $\sigma(A)$ counterclockwise avoiding the negative real axis and ζ^α takes the principal branch (see e.g. [3; 27; 32; 33; 59] and the references therein). For such operators the expected property $A^\alpha A^\beta = A^{\alpha+\beta}$ holds true. A well-known example is the fractional Laplace operator which can be also defined (in the space of Fourier transforms) as

$$\triangle^{\alpha/2} u(\mathbf{x}) = -\frac{1}{2\pi} \int_{\mathbb{R}^n} e^{-i\boldsymbol{\xi}\cdot\mathbf{x}} \|\boldsymbol{\xi}\|^\alpha\, \mathcal{F}[u](\boldsymbol{\xi})\, d\boldsymbol{\xi}, \quad \mathbf{x} \in \mathbb{R}^n. \tag{3.18}$$

The stochastic solution to the Cauchy problem involving the fractional Laplace operator (3.18) is given by the process $\mathbf{B}(\mathfrak{H}_t^\alpha)$ ($\mathbf{B}$ is a Brownian motion driven by the self-adjoint Laplace operator $\triangle$ and $\mathfrak{H}_t^\alpha$ is a stable subordinator) and has been first investigated by Bochner [11]; Feller [20]. In the one-dimensional case, the operator (3.18) becomes the Riesz operator (2.6) for which the representation (2.5) given by means of the right and left Weyl's derivatives holds as well.

In this section we study a fractional power of (3.1) which is the power of the composition of non commuting operators and obtain the explicit representation

$$\mathcal{A} f = -(-\mathcal{G}^*)^\nu f = -\frac{\partial^\nu}{\partial x^\nu}\left(x^{\mu-1+\nu}\frac{\partial^\nu}{\partial(-x)^\nu}\left(x^{1-\mu} f\right)\right) \tag{3.19}$$

defined on the positive real line $\Omega_\infty = (0, +\infty)$ for $\nu \in (0,1)$. This representation involves the Riemann-Liouville fractional derivatives (2.7) and (2.8) which replace the Weyl's derivatives (2.2) and (2.3) for functions defined on the positive real line. As we can check, from (2.9), the fractional operator (3.19) for $\nu = 1$ becomes the operator (3.1). We show that, for $\nu \in (0,1)$, the process $G_\mu(\mathfrak{H}^\nu_t)$ is the stochastic solution to the Cauchy problem involving the operator (3.19) where G_μ is driven by $\mathcal{G}^*$. First we state the following result.

Theorem 3. *Let us consider the process $G_\mu(t)$, $t>0$, satisfying the stochastic equation (3.5). For $\mu > 0$, $\forall\, \nu \in (0,1]$, the 1−dimensional density law g^1_μ of the process G_μ solves the fractional p.d.e. on $\Omega_\infty = (0,+\infty)$*

$$\frac{\partial^\nu g^1_\mu}{\partial t^\nu}(x,t) = \frac{\partial^\nu}{\partial(-x)^\nu}\left(x^{\mu-1+\nu}\frac{\partial^\nu}{\partial(-x)^\nu}\left(x^{1-\mu}\, g^1_\mu(x,t)\right)\right) \tag{3.20}$$

subject to the initial condition $g^1_\mu(x,0) = \delta(x)$.

The stochastic solution to (3.20) is the process G_μ which does not depend on the fractional index $\nu \in (0,1]$. Furthermore, we notice that the space operator appearing in (3.20) is different from $\mathcal{A}$.

We proceed our analysis by considering the process $F^{\nu,\beta}_t = \mathfrak{H}^\nu_{\mathfrak{L}^\beta_t}$, $t>0$ with law

$$\mathfrak{f}_{\nu,\beta}(x,t) = \int_0^\infty h_\nu(x,s)\, l_\beta(s,t)\, ds, \quad x \geq 0,\ t>0,\ \nu,\beta \in (0,1) \tag{3.21}$$

which has been thoroughly studied by several authors, see e.g. [16; 18; 28; 36; 38]. If $\nu = \beta$, then the law (3.21) takes the form $\mathfrak{f}_{\nu,\nu}(x,t) = t^{-1} f_\nu(t^{-1}x)$ where

$$f_\nu(x) = \frac{1}{\pi}\frac{x^{\nu-1}\sin\pi\nu}{1+2x^\nu\cos\pi\nu + x^{2\nu}}, \quad x\geq 0,\ t>0, \quad \nu\in(0,1) \tag{3.22}$$

and $F^{\nu,\nu}_t \stackrel{law}{=} t \times {}_1\mathfrak{H}^\nu_t / {}_2\mathfrak{H}^\nu_t$, $t>0$, (which means that $F^{\nu,\nu}_t \in \mathbb{P}_1$) where ${}_j\mathfrak{H}^\nu_t$, $j=1,2$ are independent stable subordinators and the ratio ${}_1\mathfrak{H}^\nu_t / {}_2\mathfrak{H}^\nu_t$ is independent of t (see [12; 16; 34; 62]). The density law f_ν arises in many important contexts, we refer to the paper by James [28] and the references therein for details.

Lemma 1. *The governing equation of the process $F^{\alpha,\beta}_t$, $t>0$, with density law (3.21), is written as*

$$\left(\frac{\partial^\beta}{\partial t^\beta} + \frac{\partial^\nu}{\partial x^\nu}\right)\mathfrak{f}_{\nu,\beta}(x,t) = \delta(x)\frac{t^{-\beta}}{\Gamma(1-\beta)}, \quad x\geq 0,\ t>0 \tag{3.23}$$

with $\mathfrak{f}_{\nu,\beta}(\partial\Omega_\infty, t) = 0$ and $\mathfrak{f}_{\nu,\beta}(x,0) = \delta(x)$ or, by considering (2.11),

$$\left({}^cD^\beta_t + \frac{\partial^\nu}{\partial x^\nu}\right)\mathfrak{f}_{\nu,\beta}(x,t) = 0, \quad x\geq 0,\ t>0 \tag{3.24}$$

with $\mathfrak{f}_{\nu,\beta}(x,0) = \delta(x)$.

Proof. From the Laplace transforms (2.13) and (2.14) we have that

$$\Psi(\xi, \lambda) = \int_0^\infty e^{-\lambda t} \left(E\, e^{-\xi F_t^{\alpha,\beta}} \right) dt = \lambda^{\beta-1}(\lambda^\beta + \xi^\nu)^{-1}.$$

Let us consider the equation (3.24). From the fact that

$$\mathcal{L}\left[{}^c D_t^\beta f \right](\lambda) = \lambda^\beta \mathcal{L}[f](\lambda) - \lambda^{\beta-1} f(0^+), \quad \beta \in (0,1)$$

(which comes from (2.11)) we obtain

$$\lambda^\beta \Psi(\xi, \lambda) - \lambda^{\beta-1} + \xi^\nu \Psi(\xi, \lambda) = 0$$

which concludes the proof. □

We state the main result of this section concerning the operator (3.19).

Theorem 4. *For $x \in \Omega_\infty = (0, +\infty)$, $t > 0$, $\mu > 0$ and $\beta, \nu \in (0, 1]$ we have that*

i) the density law $\mathfrak{g}_\mu^\nu(x,t) = \int_0^\infty g_\mu^1(x,s)\, h_\nu(s,t)\, ds$ solves the fractional p.d.e.

$$\frac{\partial \mathfrak{g}_\mu^\nu}{\partial t}(x,t) = \mathcal{A}\, \mathfrak{g}_\mu^\nu(x,t), \tag{3.25}$$

ii) the density law $\mathbf{g}_\mu^{\nu,\beta}(x,t) = \int_0^\infty g_\mu^1(x,s)\, \mathfrak{f}_{\nu,\beta}(s,t)\, ds$ solves the fractional p.d.e.

$$\frac{\partial^\beta \mathbf{g}_\mu^{\nu,\beta}}{\partial t^\beta}(x,t) = \mathcal{A}\, \mathbf{g}_\mu^{\nu,\beta}(x,t). \tag{3.26}$$

Furthermore,

$$\mathbf{g}_\mu^{\nu,\beta}(x,t) = \frac{1}{t^{\beta/\nu}} \mathbf{G}_\mu^{\nu,\beta}\left(\frac{x}{t^{\beta/\nu}} \right) \tag{3.27}$$

where

$$\mathbf{G}_\mu^{\nu,\beta}(x) = \frac{1}{x} H_{3,3}^{2,1}\left[x \middle| \begin{array}{lll} (1, \frac{1}{\nu}); & (1, \frac{\beta}{\nu}); & (\mu, 0) \\ (\mu, 1); & (1, \frac{1}{\nu}); & (1, 1) \end{array} \right], \quad x > 0$$

is a Fox's function defined in (A.13).

From Theorem 4 we have that

$$\mathfrak{g}_\mu^\nu(x,t) = E^x \delta(G_\mu(\mathfrak{H}_t^\nu)),$$

$$\mathbf{g}_\mu^{\nu,\beta}(x,t) = E^x \delta(G_\mu(F_t^{\nu,\beta}))$$

and thus $G_\mu(F_t^{\nu,\beta})$, $t > 0$ is the stochastic solution to (3.26) whereas $G_\mu(F_t^{\nu,1}) = G_\mu(\mathfrak{H}_t^\alpha)$, $t > 0$ represents the stochastic solution to (3.25). This is because of the fact that $\mathfrak{H}_{\mathfrak{L}_t^1}^\nu \stackrel{a.s.}{=} \mathfrak{H}_t^\nu$, $t > 0$, being $\mathfrak{L}_t^1 \stackrel{a.s.}{=} t$ the elementary subordinator and $l_\nu(s,t) \to \delta(t-s)$ for $\nu \to 1$. Furthermore, from the formulas (A.12) and (A.11), by taking into account (4.30), we immediately have that

$$\mathbf{G}_\mu^{\nu,\nu}(x) = \frac{1}{x} H_{2,2}^{1,1}\left[x \middle| \begin{array}{ll} (1, \frac{1}{\nu}); & (\mu, 0) \\ (\mu, 1); & (1, \frac{1}{\nu}) \end{array} \right], \quad x > 0,\ \nu \in (0,1),$$

$$\mathbf{G}_\mu^{\nu,1}(x) = \frac{1}{x} H_{2,2}^{1,1}\left[x \middle| \begin{array}{ll} (1, \frac{1}{\nu}); & (\mu, 0) \\ (\mu, 1); & (1, 1) \end{array} \right], \quad x > 0,\ \nu \in (0,1)$$

and

$$\mathbf{G}^{1,1}_{\mu}(x) = \frac{1}{x} H^{1,0}_{1,1}\left[x \middle| \begin{array}{c} (\mu, 0) \\ (\mu, 1) \end{array} \right], \quad x > 0.$$

For $\beta = \nu$, the equation (3.26) takes the form

$$\frac{\partial^\nu \mathbf{g}^{\nu,\nu}_{\mu}}{\partial t^\nu}(x,t) = -\frac{\partial^\nu}{\partial x^\nu}\left(x^{\mu-1+\nu} \frac{\partial^\nu}{\partial (-x)^\nu}\left(x^{1-\mu}\, \mathbf{g}^{\nu,\nu}_{\mu}(x,t) \right) \right) \tag{3.28}$$

which differs form (3.20) and represents the governing equation of the process

$$G_\mu \left(t \times {}_1\mathfrak{H}^\nu_1 / {}_2\mathfrak{H}^\nu{}_1 \right), \quad t > 0$$

where ${}_j\mathfrak{H}^\nu_t$, $t > 0$, $j = 1, 2$ are independent stable subordinators. Here, the stochastic solution to (3.28) depends on ν only by means of the ratio ${}_1\mathfrak{H}^\nu_1 / {}_2\mathfrak{H}^\nu_1$ which possesses distribution (3.22). For $\beta = 1$, equation (3.26) becomes (3.25) and writes

$$\frac{\partial \mathbf{g}^{\nu,1}_{\mu}}{\partial t}(x,t) = -\frac{\partial^\nu}{\partial x^\nu}\left(x^{\mu-1+\nu} \frac{\partial^\nu}{\partial (-x)^\nu}\left(x^{1-\mu}\, \mathbf{g}^{\nu,1}_{\mu}(x,t) \right) \right) \tag{3.29}$$

($\mathfrak{g}^\nu_\mu = \mathbf{g}^{\nu,1}_\mu$) with stochastic solution given by $G_\mu(\mathfrak{H}^\nu_t)$, $\quad t > 0$. Finally, for $\nu = 1$ in (3.26), we reobtain the governing equation of the process $G_\mu(L^\nu_t)$, $t > 0$ already investigated in Theorem 1.

Remark 2. We observe that, for $\mu \in \mathbb{N}$,

$$G_\mu(\mathfrak{H}^\nu_t) = \|\mathbf{B}(\mathfrak{H}^\nu_t)\|^2 = \sum_{j=1}^{\mu} [{}_jB(\mathfrak{H}^\nu_t)]^2$$

where $\mathbf{B}(t) = ({}_1B(t), \dots, {}_nB(t))$, $t > 0$ and ${}_jB(t)$, $j = 1, \dots, n$ are independent Brownian motions. From the Bochner's subordination rule we get ${}_jB(\mathfrak{H}^\nu_t) \stackrel{law}{=} {}_jS^{2\nu}_t$ which are symmetric stable processes with $E \exp i\xi {}_jS^{2\nu}_t = \exp -t|\xi|^{2\nu}$ for all $j = 1, \dots, n$. Thus, by considering n independent stable processes ${}_jS^{2\nu}_t$ and $\mathbf{S}^{2\nu}_t = ({}_1S^{2\nu}_t, \dots, {}_nS^{2\nu}_t)$, $t > 0$, we obtain that $G_\mu(\mathfrak{H}^\nu_t) \stackrel{law}{=} \|\mathbf{S}^{2\nu}_t\|^2$.

Remark 3. For $\mathbf{B}(t) \in \mathbb{R}^3$, the three dimensional Bessel process represents a radial diffusion on a homogeneous ball. The subordinated squared Bessel process can be therefore regarded as a radial diffusion on a non-homogeneous ball, with fractal structure for instance. This interpretation is due to the fact that the random time $\mathfrak{H}^\nu_t$ has non-negative increments and therefore, non-decreasing paths. Furthermore, the subordinated process $G_\mu(\mathfrak{H}^\nu_t)$ speed up as $\mathfrak{H}^\nu_t$ increases. For $\nu \to 1$, $\mathfrak{H}^\nu_t \to t$ a.s. and $G_\mu(\mathfrak{H}^\nu_t)$ becomes standard diffusion because of the linear growing of time.

Remark 4. From (4.31) we obtain that

$$E\left[G_\mu(F^{\nu,\beta}_t) \right]^r \propto t^{\frac{\beta}{\nu} r}, \quad r > 0,\ \nu, \beta \in (0, 1].$$

Moreover, $F^{\nu,\beta}_t \to \mathfrak{L}^\beta_t$ for $\nu \to 1$ and therefore we have that $G_\mu(\mathfrak{L}^\beta_t)$ is a subdiffusion whereas, from the fact that $F^{\nu,\beta}_t \to \mathfrak{H}^\nu_t$ for $\beta \to 1$ we get the superdiffusion $G_\mu(\mathfrak{H}^\nu_t)$.

3.3 Explicit representations of solutions via Mellin convolutions

The laws of the processes $\mathfrak{H}_t^\nu$ and $\mathfrak{L}_t^\nu$ can be written in terms of H-functions as pointed out in Section 2. Alternative expressions can be given in terms of the Wright function but only for $\nu = 1/2, 1/3$ we obtain a closed form of the density laws and therefore of the solutions investigated in the previous section.

We give an explicit representation of the solutions presented so far by exploiting the Mellin convolutions of generalized gamma functions. The simplest convolutions we deal with are written below: for $x, t > 0$ and $\gamma \neq 0$, μ_1, $\mu_2 > 0$,

$$g_{\mu_1}^{\gamma} \star g_{\mu_2}^{-\gamma}(x,t) = \frac{|\gamma|}{B(\mu_1,\mu_2)} \frac{x^{\gamma\mu_1 - 1} t^{\gamma\mu_2}}{(t^\gamma + x^\gamma)^{\mu_1+\mu_2}} \tag{3.30}$$

where $B(\cdot,\cdot)$ is the Beta function (see e.g. Gradshteyn and Ryzhik [25, formula 8.384]) and,

$$g_{\mu_1}^{\gamma} \star g_{\mu_2}^{\gamma}(x,t) = \frac{2|\gamma| \, (x^\gamma/t^\gamma)^{\frac{\mu_1+\mu_2}{2}}}{x\,\Gamma(\mu_1)\Gamma(\mu_2)} K_{\mu_2-\mu_1}\left(2\sqrt{(x^\gamma/t^\gamma)}\right), \tag{3.31}$$

where K_α is the modified Bessel function of the second kind. In particular

$$K_\alpha(z) = \frac{\pi}{2} \frac{I_{-\alpha}(z) - I_\alpha(z)}{\sin \alpha\pi}, \quad \alpha \text{ not integer} \tag{3.32}$$

(see [25, formula 8.485]) where

$$I_\alpha(z) = \sum_{k\geq 0} \frac{(z/2)^{\alpha+2k}}{k!\,\Gamma(\alpha+k+1)}, \quad |z| < \infty, \quad |\arg z| < \pi \tag{3.33}$$

is the modified Bessel function of the first kind (see [25, formula 8.445]).

Definition 1. *For $-\infty < a < b < \infty$, we define the space*

$$\mathbb{M}_a^b = \left\{ f : \mathbb{R}_+ \mapsto \mathbb{C} \,\middle|\, x^{\eta-1} f(x) \in L^1(\mathbb{R}_+), \forall \eta \in \mathbb{H}_a^b \right\}$$

where $\mathbb{H}_a^b = \{\zeta : \zeta \in \mathbb{C}, a < \Re\{\zeta\} < b\}$.

Definition 2. *Let us consider the function $\mathcal{P}_Y : \mathbb{C} \mapsto \mathbb{R}$. We define the class of one-dimensional processes*

$$\mathbb{P}_\alpha = \left\{ Y(t),\, t > 0 \,:\, \exists \mathbb{S} \subset \mathbb{H}_a^b \text{ s.t. } E\big|Y(t^\alpha)/t\big|^{\eta-1} = \mathcal{P}_Y(\eta),\ \forall \eta \in \mathbb{S} \right\}, \quad \alpha \in \mathbb{R}.$$

Remark 5. We notice that $S_t^\nu \in \mathbb{P}_\nu \;\Leftrightarrow\; S_t^\nu \overset{law}{=} t^{1/\nu} S_1^\nu$.

Definition 3. *We define the class of functions*

$$\mathbb{F}_\alpha = \left\{ f \,\middle|\, Y \sim f,\, Y \in \mathbb{P}_\alpha \right\}$$

where $Y \sim f$ means that the process Y possesses the density law f.

Remark 6. We remark that $\mathbb{F}_\alpha \subset \mathbb{M}_a^b$.

We point out that for a composition involving the processes $Y_{\sigma_j} = \left[X_{\sigma_j}\right]^\alpha$ where X_{σ_j} are independent processes such that $X_{\sigma_j} \in \mathbb{P}_\alpha$ for all $j = 1, 2, \ldots n$, we have that $Y_{\sigma_j} \in \mathbb{P}_1$, $\forall j$ which implies that

$$Y_{\sigma_1}(Y_{\sigma_2}(\ldots Y_{\sigma_n}(t)\ldots)) \overset{law}{=} Y_{\sigma_1}(t^{1/n}) Y_{\sigma_2}(t^{1/n}) \cdots Y_{\sigma_n}(t^{1/n}) \tag{3.34}$$

for all possible permutations of $\{\sigma_j\}$, $j = 1, 2, \dots, n$. This can be easily carried out by observing that $\mathcal{P}_{Y_{\sigma_j}}(\eta) = \mathcal{P}_{X_{\sigma_j}}(\eta\alpha - \alpha + 1)$. Indeed,

$$E[X_{\sigma_j}(t)]^{\eta-1} = \mathcal{P}_{X_{\sigma_j}}(\eta)\, t^{\frac{\eta-1}{\alpha}} \quad \Rightarrow \quad E[Y_{\sigma_j}(t)]^{\eta-1} = E[X_{\sigma_j}]^{(\eta\alpha-\alpha+1)-1}$$

and therefore, for $i \neq j$,

$$E[Y_{\sigma_j}(Y_{\sigma_i}(t))]^{\eta-1} = \mathcal{P}_{X_{\sigma_j}}(\eta\alpha - \alpha + 1) E[Y_{\sigma_i}(t)]^{\eta-1} = E[Y_{\sigma_j}(t^{1/2})\, Y_{\sigma_i}(t^{1/2})]^{\eta-1}.$$

From the fact that $\mathbb{P}_1 \ni G_\mu \sim g^1_\mu \in \mathbb{F}_1$ and

$$\mathbb{P}_\gamma \ni [G_\mu(t)]^{1/\gamma} \sim \tilde{g}^\gamma_\mu \quad \Leftrightarrow \quad \tilde{g}^\gamma_\mu \in \mathbb{F}_\gamma$$

$$\mathbb{P}_1 \ni [G_\mu(t^\gamma)]^{1/\gamma} \sim g^\gamma_\mu \quad \Leftrightarrow \quad g^\gamma_\mu \in \mathbb{F}_1$$

we can define the following Mellin convolution of $g^\gamma_\mu \in \mathbb{M}^\infty_{1-\gamma\mu}$.

Definition 4. *For $\gamma \neq 0$, $\boldsymbol{\mu} = (\mu_1, \dots, \mu_n) \in \mathbb{R}^n_+$ we define the Mellin convolution*

$$g^{\gamma,\star n}_{\boldsymbol{\mu}}(x,t) = g^\gamma_{\mu_1} \star \dots \star g^\gamma_{\mu_n}(x,t) \tag{3.35}$$

with Mellin transform (see (A.1))

$$\mathcal{M}[g^{\gamma,\star n}_{\boldsymbol{\mu}}(\cdot,t)](\eta) = \prod_{j=1}^n \mathcal{M}[g^\gamma_{\mu_j}(\cdot, t^{1/n})](\eta) = t^{\eta-1} \prod_{j=1}^n \frac{\Gamma((\eta-1)/\gamma + \mu_j)}{\Gamma(\mu_j)} \tag{3.36}$$

where $\eta \in \mathbb{H}^1_a$ and $a = 1 - \min_j\{\gamma\mu_j\}$.

We notice that $f_{X^\alpha} \star f_{Y^\beta} = f_{Y^\beta} \star f_{X^\alpha}$ is the law of $X^\alpha \cdot Y^\beta$ if $X \in \mathbb{P}_\alpha$ and $Y \in \mathbb{P}_\beta$ whereas the well-known Fourier convolution $f_X * f_Y$ is the law of $X + Y$. Also, we introduce the sets

$$\mathscr{S}^n_\kappa(\varsigma) = \left\{ \bar{\wp} \in \mathbb{R}^n_+ \,:\, \bar{\wp} = \frac{\bar{\upsilon}}{\kappa},\, \bar{\upsilon} = (\upsilon_1, \dots, \upsilon_n) \in \mathbb{N}^n,\, \sum_{j=1}^n \upsilon_j = \varsigma \right\} \tag{3.37}$$

and

$$\mathscr{P}^n_\kappa(\varrho) = \left\{ \bar{\wp} \in \mathbb{R}^n_+ \,:\, \bar{\wp} = \frac{\bar{\upsilon}}{\kappa},\, \bar{\upsilon} = (\upsilon_1, \dots, \upsilon_n) \in \mathbb{N}^n,\, \prod_{j=1}^n \upsilon_j = \varrho \right\} \tag{3.38}$$

with $m, \kappa, \varrho \in \mathbb{N}$. For $\gamma = 1, 2$ and a fixed $\boldsymbol{\mu} = (\mu_1, \dots, \mu_n) \in \mathscr{S}^n_\kappa(\varsigma)$, we have that

$$g^\gamma_{\mu_1} * \dots * g^\gamma_{\mu_n}(x,t) = g^\gamma_{\theta_{\sigma_1}} * \dots * g^\gamma_{\theta_{\sigma_n}}(x,t)$$

for all $\boldsymbol{\theta} = (\theta_{\sigma_1}, \dots, \theta_{\sigma_n}) \in \mathscr{S}^n_\kappa(\varsigma)$ and all permutations of $\{\sigma_j\}$, $j = 1, 2, \dots, n$. This fact follows easily from the semigroup property ($*$-commutativity) of the law g^γ_μ which will be shown in *ii*), Lemma 5 below. We observe that $\aleph = |\mathscr{P}^m_\kappa| < |\mathbb{N}|$ is the cardinality of $\mathscr{P}^m_\kappa$, thus $\mathscr{P}^m_\kappa$ is a finite set. Furthermore, $\forall \varrho \in \mathbb{N}$ and a fixed $\boldsymbol{\mu} \in \mathscr{P}^m_\kappa(\varrho)$, we have that

$$\mathcal{M}[g^{\gamma,\star n}_{\boldsymbol{\mu}}(\cdot,t)](\eta) = \mathcal{M}[g^{\gamma,\star n}_{\boldsymbol{\theta}}(\cdot,t)](\eta), \quad \forall \boldsymbol{\theta} \in \mathscr{P}^m_\kappa(\varrho) \tag{3.39}$$

whereas, for $\boldsymbol{\mu} \in \mathscr{S}^m_\kappa(\varsigma)$ and $\gamma = 1, 2$, we have that

$$\mathcal{F}[g^{\gamma,*n}_{\boldsymbol{\mu}}(\cdot,t)](\xi) = \mathcal{F}[g^{\gamma,*n}_{\boldsymbol{\theta}}(\cdot,t)](\xi), \quad \forall \boldsymbol{\theta} \in \mathscr{S}^m_\kappa(\varsigma) \tag{3.40}$$

where we used, the familiar notation, $f^{*n}_{\boldsymbol{\mu}} = f_{\mu_1} * \dots * f_{\mu_n}$. The symbols $\mathcal{M}$ and $\mathcal{F}$ stand for the Mellin and Fourier transforms.

Theorem 5. *Let us consider $\nu = 1/(n+1)$, $n \in \mathbb{N}$ and $\boldsymbol{\mu} = (\mu_1, \dots, \mu_n)$.*

i) For the stable subordinator $\mathfrak{H}_t^\nu$, $t > 0$, the following equivalence in law holds true

$$\mathfrak{H}_t^\nu \stackrel{law}{=} E_{\mu_1}(E_{\mu_2}(\dots E_{\mu_n}((\nu t)^{1/\nu})\dots)), \quad t > 0, \quad \boldsymbol{\mu} \in \mathscr{P}_{n+1}^n(n!) \tag{3.41}$$

where the process $E_\mu(t)$, $t > 0$, satisfies the SDE (3.6).

ii) For the inverse process $\mathfrak{L}_t^\nu$, $t > 0$, the following equivalence in law holds true

$$\mathfrak{L}_t^\nu \stackrel{law}{=} \left[G_{\mu_1}(G_{\mu_2}(\dots G_{\mu_n}(t^\nu/\nu)\dots))\right]^\nu, \quad t > 0, \quad \boldsymbol{\mu} \in \mathscr{P}_{n+1}^n(n!). \tag{3.42}$$

where the process $G_\mu(t)$, $t > 0$, satisfies the SDE (3.5).

From Theorem 5 and formulas (3.30), (3.31) we can explicitly write $\mathbf{g}_\mu^{\nu,\beta}$. Furthermore, this representation holds in the general set $\mathscr{P}_{n+1}^n(n!)$. Indeed, for $\nu = 1/(n+1)$, $n \in \mathbb{N}$, the stochastic solution to (3.25) is given by

$$G_\mu(E_{\mu_1}(E_{\mu_2}(\dots E_{\mu_n}((\nu t)^{1/\nu})\dots)), \quad t > 0, \quad (\mu_1, \dots, \mu_n) \in \mathscr{P}_{n+1}^n(n!).$$

Moreover, the representation (3.42) turns out to be useful in representing the solution to the problems (3.3) and (3.13). A natural extension follows for the problem (3.26). From $E_\mu(ct) \stackrel{law}{=} \frac{1}{c} E_\mu$, $t > 0$, $c > 0$ and formula (3.41) we obtain that

$$\mathfrak{H}_t^\nu \stackrel{law}{=} \nu^{-1/\nu} E_{\mu_1}(E_{\mu_2}(\dots E_{\mu_n}(t^{1/\nu})\dots)), \quad t > 0$$

and thus, for $\alpha_1 = 1/(n_1+1)$, $\alpha_2 = 1/(n_2+1)$ and

$$\boldsymbol{\mu}_1 = (\mu_{1,1}, \dots, \mu_{1,n_1}) \in \mathscr{P}_{n_1+1}^{n_1}(n_1!), \qquad \boldsymbol{\mu}_2 = (\mu_{2,1}, \dots, \mu_{2,n_2}) \in \mathscr{P}_{n_2+1}^{n_2}(n_2!)$$

we have that

$$F_t^{\alpha_1,\alpha_2} \stackrel{law}{=} \nu^{-1/\nu} E_{\mu_{1,1}}(E_{\mu_{1,2}}(\dots E_{\mu_{1,n_1}}(G_{\mu_{1,2}}(G_{\mu_{2,2}}(\dots G_{\mu_{2,n_2}}(t^\nu/\nu)\dots)))\dots)). \tag{3.43}$$

The fact that $E_\mu \in \mathbb{P}_1$ and $G_\mu \in \mathbb{P}_1$ means that

$$E_{\mu_1}(G_{\mu_2}(t)) \stackrel{law}{=} G_{\mu_2}(E_{\mu_1}(t))$$

or equivalently

$$g_{\mu_1}^{-1} \star g_{\mu_2}^1 = g_{\mu_2}^1 \star g_{\mu_1}^{-1}.$$

From this we can write the law (3.21) in terms of the convolutions (3.30) and (3.31).

Corollary 1. *For $\nu = 1/(n+1)$, $n \in \mathbb{N}$, the stochastic solution to (3.26) is given by*

$$G_\mu(F_t^{\nu,\beta}), \quad t > 0$$

where G_μ has law g_μ^1 and $F^{\nu,\beta}$ has density which can be represented by means of the Mellin convolutions (3.30) and (3.31) as formula (3.43) entails.

Remark 7. For $\gamma \neq 0$, $\mu > 0$ and $\nu = 1/5$, the stochastic solution to (3.3) can be written as follows

$$\left[G_\mu(G_{\mu_1}(G_{\mu_2}(G_{\mu_3}(G_{\mu_4}(t^{1/5}/5)))))\right]^{1/\gamma}, \quad t > 0$$

($G_\mu, G_{\mu_1}, \dots, G_{\mu_4}$ independent squared Bessel processes) where

$$\boldsymbol{\mu} = (\mu_1, \mu_2, \mu_3, \mu_4) \in \mathscr{P}_5^4(4!).$$

For $\boldsymbol{\mu} = (3, 2, 2, 2)/5$, we have that

$$\tilde{u}^{\gamma,\mu}_{1/5}(x,t) = C\, \frac{x^{\gamma\mu-1}}{t^{2/5}} \int_0^\infty \int_0^\infty \frac{e^{-x^\gamma/z}}{s^{2/5}} K_{\frac{1}{5}}\left(2\frac{z^{5/2}}{s}\right) K_0\left(\frac{2}{5^{3/2}}\frac{s}{t^{1/2}}\right) ds\, dz$$

where

$$C = \frac{2\,\Gamma\left(\frac{1}{5}\right)\Gamma\left(\frac{4}{5}\right)}{5^{3/2}\left[\pi\Gamma\left(\frac{2}{5}\right)\right]^2\Gamma\left(\frac{3}{2}\right)}.$$

For further configurations of $\boldsymbol{\mu}$, see Remark 9.

Remark 8. The relation between stable densities and higher order equations has been investigated by many authors (see for example Baeumer et al. [2]; DeBlassie [15]; D'Ovidio [17]; D'Ovidio and Orsingher [18]). In [17] we have shown that the law $l_{\frac{1}{n}}$ of $\mathfrak{L}_t^{\frac{1}{n}}$ solves the higher-order equation

$$(-1)^n \frac{\partial^n u}{\partial x^n} = \frac{\partial u}{\partial t}.$$

4 Auxiliary Results and Proofs

4.1 The operators $\mathcal{G}$ and $\mathcal{G}^*$

The operators we deal with are given by

$$\begin{aligned}\mathcal{G}^* f_2 =& \frac{1}{\gamma^2}\left(\frac{\partial}{\partial x}x^{2-\gamma}\frac{\partial}{\partial x} - (\gamma\mu - 1)\frac{\partial}{\partial x}x^{1-\gamma}\right) f_2\\ =& \frac{1}{\gamma^2}\frac{\partial}{\partial x}\left(x^{\gamma\mu-\gamma+1}\frac{\partial}{\partial x}\left(\frac{1}{\mathfrak{w}(x)} f_2\right)\right), \quad f_2 \in D(\mathcal{G}^*)\end{aligned} \tag{4.1}$$

and

$$\begin{aligned}\mathcal{G} f_1 =& \frac{x^{1-\gamma}}{\gamma^2}\left(x\frac{\partial^2}{\partial x^2} + (\gamma\mu - \gamma + 1)\frac{\partial}{\partial x}\right) f_1\\ =& \frac{1}{\gamma^2\,\mathfrak{w}(x)}\frac{\partial}{\partial x}\left(x^{\gamma\mu-\gamma+1}\frac{\partial}{\partial x} f_1\right), \quad f_1 \in D(\mathcal{G})\end{aligned} \tag{4.2}$$

where $\mathfrak{w}(x) = x^{\gamma\mu-1}$. We shall refer to $\mathcal{G}^*$ as the adjoint of $\mathcal{G}$. Indeed, as a straightforward check shows, we have that $\mathcal{G}^*\,\mathfrak{w}\, f_1 = \mathfrak{w}\,\mathcal{G}\, f_1$ and the Lagrange's identity

$$f_2\,\mathcal{G}\, f_1 - f_1\,\mathcal{G}^*\, f_2 = 0 \tag{4.3}$$

immediately follows. Thus, by observing that

$$D(\mathcal{G}^*) = \{f \in \tilde{\mathbb{M}}_1 : f = \mathfrak{w}\, f_1,\ f_1 \in D(\mathcal{G})\},$$

(see Definition 1) we obtain that

$$\langle \mathcal{G} f_1,\ f_2\rangle = \langle f_1,\ \mathcal{G}^* f_2\rangle, \quad \forall f_1 \in D(\mathcal{G}) \text{ and } \forall f_2 \in D(\mathcal{G}^*).$$

Lemma 2. *The following hold true:*

i) For the operator appearing in (4.2) *we have that*

$$\mathcal{G}\,\psi_\kappa = (\kappa/2)^2\,\psi_\kappa \tag{4.4}$$

where

$$\psi_\kappa(x) = x^{\frac{\gamma}{2}(1-\mu)} K_{\mu-1}\left(\kappa\, x^{\gamma/2}\right), \quad \kappa > 0,\ x > 0,\ \gamma \neq 0 \tag{4.5}$$

and K_α is the Macdonald's function (3.32).

ii) For the operator appearing in (4.2) *we have that*

$$\mathcal{G}\,\bar{\psi}_\kappa = -(\kappa/2)^2\,\bar{\psi}_\kappa \tag{4.6}$$

where

$$\bar{\psi}_\kappa(x) = x^{\frac{\gamma}{2}(1-\mu)} J_{\mu-1}\left(\kappa\, x^{\gamma/2}\right), \quad \kappa > 0,\ x > 0,\ \gamma \neq 0 \tag{4.7}$$

and J_α is the Bessel function of the first kind (3.12).

Proof. We first recall some properties of the Macdonald's function (3.32): we will use the fact that $K_{-\alpha} = K_\alpha$ and

$$\frac{d}{dz} K_\alpha(z) = -K_{\alpha-1}(z) - \frac{\alpha}{z} K_\alpha(z). \tag{4.8}$$

(see [35, p. 110]); the functions K_α and I_α are two linearly independent solutions of the Bessel equation

$$x^2 \frac{d^2 Z_\alpha(x)}{dx^2} + x\frac{dZ_\alpha(x)}{dx} - x^2 Z_\alpha(x) = 0 \tag{4.9}$$

whereas, the functions J_α and Y_α (see [35] for definition) are linearly independent solutions to

$$x^2 \frac{d^2 Z_\alpha(x)}{dx^2} + x\frac{dZ_\alpha(x)}{dx} + x^2 Z_\alpha(x) = 0 \tag{4.10}$$

(see [35, pp. 105 - 110]).

By performing the first and the second derivative with respect to x of the function $\psi_\kappa = \psi_\kappa(x)$ we obtain

$$\begin{aligned}\psi'_\kappa =& \frac{\gamma}{2}(1-\mu)\frac{1}{x}\psi_\kappa + x^{\frac{\gamma}{2}(1-\mu)}\frac{\gamma\kappa}{2x}x^{\gamma/2}\left[-K_{-\mu} - \frac{1-\mu}{\kappa x^{\gamma/2}}K_{1-\mu}\right]\\ =& \frac{\gamma}{2}(1-\mu)\frac{1}{x}\psi_\kappa - \frac{\gamma\kappa}{2x}x^{\frac{\gamma}{2}(2-\mu)}K_{-\mu} - \frac{\gamma}{2}(1-\mu)\frac{1}{x}\psi_\kappa = -\frac{\gamma\kappa}{2x}x^{\frac{\gamma}{2}(2-\mu)}K_{-\mu}\end{aligned}$$

and

$$\begin{aligned}\psi''_\kappa =& \left(\frac{\gamma}{2}(2-\mu) - 1\right)\frac{1}{x}\psi'_\kappa + \frac{\gamma\kappa}{2}x^{\frac{\gamma}{2}(2-\mu)-1}\frac{\gamma\kappa}{2x}x^{\gamma/2}\left[-K_{\mu-1} - \frac{\mu}{\kappa x^{\gamma/2}}K_\mu\right]\\ =& \left(\frac{\gamma}{2}(2-\mu) - 1\right)\frac{1}{x}\psi'_\kappa - \left(\frac{\gamma\kappa}{2}\right)^2\frac{x^{\frac{\gamma}{2}(1-\mu)+\gamma}}{x^2}K_{\mu-1} - \frac{\gamma\mu}{2x}\psi'_\kappa.\end{aligned}$$

By keeping in mind the operator $\mathcal{G}$, from the fact that

$$x\psi''_\kappa + (\gamma\mu - \gamma + 1)\psi'_\kappa = x^{\gamma-1}\frac{\gamma^2\kappa^2}{2^2}\psi_\kappa \tag{4.11}$$

the relation (4.4) is obtained. The equation (4.11) can be rewritten as

$$x^2\,\psi''_\kappa + (\gamma\mu - \gamma + 1)\,x\,\psi'_\kappa - \gamma^2\,(\kappa/2)^2\,x^\gamma\,\psi_\kappa = 0 \tag{4.12}$$

which is related to the formula (4.9) whereas, a slightly modified version of (4.12), which is

$$x^2\,\bar\psi''_\kappa + (\gamma\mu - \gamma + 1)\,x\,\bar\psi'_\kappa + \gamma^2\,(\kappa/2)^2\,x^\gamma\,\bar\psi_\kappa = 0, \tag{4.13}$$

is related to the formula (4.10). The equation (4.6) can be written as formula (4.13) and therefore, after some algebra, from (4.10), we have at once that

$$\bar\psi_\kappa(x) = x^{\frac{\gamma}{2}(1-\mu)}\,J_{\mu-1}\left(\kappa\,x^{\gamma/2}\right)$$

as announced in the statement of the Lemma. □

Formula (4.13) can be put into the Sturm-Liouville form as follows

$$\left(x^{\gamma\mu-\gamma+1}\bar\psi'_\kappa\right)' + \gamma^2\,(\kappa/2)^2\,\mathfrak{w}(x)\,\bar\psi_\kappa = 0. \tag{4.14}$$

According to the Sturm-Liouville theory ([14]) and formula (4.14), we obtain an orthogonal system $\{\bar\psi_{\kappa_i}\}_{i\in\mathbb{N}}$ such that

$$\mathcal{G}\,\bar\psi_{\kappa_i} = -(\kappa_i/2)^2\,\bar\psi_{\kappa_i}, \tag{4.15}$$

where κ_i are the zeros of J_α and $\mathcal{G}$ is a Hermitian linear operator whose eigenfunctions are orthogonal w.r.t. the weight function $\mathfrak{w}(x) = x^{\gamma\mu-1}$. Indeed, from the fact that

$$\int_0^1 x\,J_\nu(\kappa_i x)\,J_\nu(\kappa_j x)\,dx = 0, \quad \text{if } \ i \neq j$$

(see [35]) we get that

$$\int_0^1 \bar\psi_{\kappa_i}(x)\bar\psi_{\kappa_j}(x)\mathfrak{w}(x)dx = 0, \quad \text{if } \ i \neq j. \tag{4.16}$$

4.2 Proof of Theorem 1 : time-fractional diffusions in one-dimensional half-space.

For $\nu = 1$ the density law (3.4) becomes the law of G_μ, $\tilde u_1^{\gamma,\mu} = \tilde g_\mu^\gamma$ whose Mellin transform is written as

$$\Psi_t(\eta) = \mathcal{M}[\tilde g_\mu^\gamma(\cdot,t)](\eta) = \Gamma\left(\frac{\eta-1}{\gamma}+\mu\right)\frac{t^{\frac{\eta-1}{\gamma}}}{\Gamma(\mu)}, \quad \eta \in \mathbb{H}^\infty_{1-\gamma\mu}. \tag{4.17}$$

We perform the time derivative of (4.17) and obtain

$$\begin{aligned}
\frac{\partial}{\partial t}\Psi_t(\eta) =& \frac{\eta-1}{\gamma}\Gamma\left(\frac{\eta-1}{\gamma}+\mu\right)t^{\frac{\eta-\gamma-1}{\gamma}}\\
=& \frac{\eta-1}{\gamma}\left(\frac{\eta-\gamma-1+\gamma\mu}{\gamma}\right)\Gamma\left(\frac{\eta-\gamma-1}{\gamma}+\mu\right)t^{\frac{\eta-\gamma-1}{\gamma}}\\
=& \frac{1}{\gamma^2}(\eta-1)(\eta-\gamma-1+\gamma\mu)\Psi_t(\eta-\gamma)
\end{aligned}$$

$$
\begin{aligned}
=&\frac{1}{\gamma^2}(\eta-1)(\eta-\gamma)\Psi_t(\eta-\gamma)+\frac{1}{\gamma^2}(\eta-1)(\gamma\mu-1)\Psi_t(\eta-\gamma)\\
=&\frac{1}{\gamma^2}\mathcal{M}\left[\frac{\partial}{\partial x}x^{2-\gamma}\frac{\partial}{\partial x}\tilde{g}^\gamma_\mu\right](\eta)-\frac{(\gamma\mu-1)}{\gamma^2}\mathcal{M}\left[\frac{\partial}{\partial x}x^{1-\gamma}\tilde{g}^\gamma_\mu\right](\eta).
\end{aligned}
$$

From the fact that $\tilde{g}^\gamma_\mu \in \tilde{\mathbb{M}}_1$ and according to the properties (A.8), (A.2) and (A.3), the inverse Mellin transform yields the claimed result. We give an alternative proof by exploiting the Laplace transform technique. The Laplace transform of $\tilde{g}^\gamma_\mu(x,t)$, $x,t>0$, can be evaluated by recalling that (see [25, formula 3.478])

$$
\int_0^\infty x^{\nu-1}\exp\left\{-\beta x^p-\gamma x^{-p}\right\}dx=\frac{2}{p}\left(\frac{\gamma}{\beta}\right)^{\frac{\nu}{2p}}K_{\frac{\nu}{p}}\left(2\sqrt{\gamma\beta}\right) \tag{4.18}
$$

where $p,\gamma,\beta,\nu>0$ and K_ν is the modified Bessel function. Thus, we obtain

$$
\mathcal{L}[\tilde{g}^\gamma_\mu(x,\cdot)](\lambda)=2\frac{x^{\frac{\gamma}{2}(\mu+1)-1}}{\Gamma(\mu)\lambda^{\frac{1-\mu}{2}}}K_{1-\mu}\left(2\lambda^{1/2}x^{\gamma/2}\right)=2\frac{\mathfrak{w}(x)}{\Gamma(\mu)}f(\lambda)\,\psi(x;2\lambda^{1/2})
$$

where $f(\lambda)=\lambda^{(\mu-1)/2}$ and $\psi_\kappa(x)=\psi(x;\kappa)$ is that in Lemma 2. By considering that $\tilde{g}^\gamma_\mu(x,t)=\mathfrak{w}(x)\tilde{k}^\gamma_\mu(x,t)$ and $\mathcal{G}^*\,\mathfrak{w}(x)\tilde{k}^\gamma_\mu=\mathfrak{w}(x)\,\mathcal{G}\,\tilde{k}^\gamma_\mu$ we get that

$$
\mathcal{L}[\mathcal{G}^*\,\tilde{g}^\gamma_\mu(x,\cdot)](\lambda)=2\frac{\mathfrak{w}(x)}{\Gamma(\mu)}f(\lambda)\,\mathcal{G}\,\psi(x;2\lambda^{1/2})=\lambda\,\mathcal{L}[\tilde{g}^\gamma_\mu(x,\cdot)](\lambda)
$$

where in the last formula we used the result (4.4). From the fact that

$$
\mathcal{L}\left[\frac{\partial}{\partial t}\tilde{g}^\gamma_\mu(x,\cdot)\right](\lambda)=\lambda\mathcal{L}[\tilde{g}^\gamma_\mu(x,\cdot)](\lambda),\quad x>0
$$

we obtain the claimed result for $\nu=1$.

Now, we consider $\nu\in(0,1)$. From the Laplace transform

$$
\mathcal{L}[l_\nu(x,\cdot)](\lambda)=\lambda^{\nu-1}\exp(-x\lambda^\nu)
$$

(see formula (2.14)) we obtain that

$$
\begin{aligned}
\mathcal{L}[\tilde{u}^{\gamma,\mu}_\nu(x,\cdot)](\lambda)=&\int_0^\infty \tilde{g}^\gamma_\mu(x,s)\,\mathcal{L}[l_\nu(s,\cdot)](\lambda)\,ds\\
=&2\frac{\mathfrak{w}(x)}{\Gamma(\mu)}\frac{\lambda^{\nu-1}}{\lambda^{\frac{\nu}{2}(1-\mu)}}x^{\frac{\gamma}{2}(1-\mu)}K_{1-\mu}\left(2x^{\gamma/2}\lambda^{\nu/2}\right)\\
=&2\frac{\mathfrak{w}(x)}{\Gamma(\mu)}f(\lambda)\,\psi_\kappa(x)
\end{aligned}
$$

where $\psi_\kappa(x)=\psi(x;\kappa)$ is that in (4.7) with $\kappa=2\lambda^{\nu/2}$ and $f(\lambda)=\lambda^{\nu(\mu+1)/2-1}$. Thus, in the right-hand side of (3.3) we obtain

$$
\mathcal{L}[\mathcal{G}^*\,\tilde{u}^{\gamma,\mu}_\nu(x,\cdot)](\lambda)=\frac{2}{\Gamma(\mu)}f(\lambda)\,\mathcal{G}^*\,\mathfrak{w}(x)\,\psi(x;2\lambda^{\nu/2})=2\frac{\mathfrak{w}(x)}{\Gamma(\mu)}f(\lambda)\,\mathcal{G}\,\psi(x;2\lambda^{\nu/2})
$$

where we have used the fact that $\mathcal{G}^*\,\mathfrak{w}\,f=\mathfrak{w}\,\mathcal{G}\,f$. Finally, from (4.4), we obtain

$$
\mathcal{L}[\mathcal{G}^*\,\tilde{u}^{\gamma,\mu}_\nu(x,\cdot)](\lambda)=\lambda^\nu\,\mathcal{L}[\tilde{u}^{\gamma,\mu}_\nu(x,\cdot)](\lambda). \tag{4.19}
$$

We note that $|\tilde{u}^{\gamma,\mu}_\nu(\cdot,t)|\leq Be^{-q_0 t}$ for some $B,q_0>0$ as a function of t and thus,

$$
\mathcal{L}\left[\frac{\partial^\nu\tilde{u}^{\gamma,\mu}_\nu}{\partial t^\nu}(x,\cdot)\right](\lambda)=\lambda^\nu\mathcal{L}[\tilde{u}^{\gamma,\mu}_\nu(x,\cdot)](\lambda), \tag{4.20}
$$

see [29, Lemma 2.14]. By comparing (4.19) with (4.20) the result follows.

4.3 Proof of Theorem 2 : regular Sturm-Liouville problems.

From the fact that $\mathcal{G}^* \, m_\nu^{\gamma,\mu}(x,t) = \mathfrak{w}(x)\mathcal{G}\,\bar{m}_\nu^{\gamma,\mu}(x,t)$, the problem (3.13) reduces to

$$\frac{\partial^\nu}{\partial t^\nu}\,\bar{m}_\nu^{\gamma,\mu} = \mathcal{G}\,\bar{m}_\nu^{\gamma,\mu}, \quad m_\nu^{\gamma,\mu}(\partial\Omega_1,t)=0, \quad \bar{m}_\nu^{\gamma,\mu}(x,0)=m_0(x)/\mathfrak{w}(x) \tag{4.21}$$

where $m_\nu^{\gamma,\mu}(x,t) = \mathfrak{w}(x)\,\bar{m}_\nu^{\gamma,\mu}(x,t)$ and $\mathfrak{w}(x) = x^{\gamma\mu-1}$. Furthermore, from Lemma 2, we have that $\mathcal{G}\,\bar{\psi}_{\kappa_i} = -(\kappa_i/2)^2\bar{\psi}_{\kappa_i}$ where

$$\bar{\psi}_{\kappa_i}(x) = x^{\frac{\gamma}{2}(1-\mu)} J_{\mu-1}\left(\kappa_i x^{\gamma/2}\right) \tag{4.22}$$

and κ_i, $i\in\mathbb{N}$ are the zeros of J_α. Formula (4.16) leads to the orthonormal system $\left\{\bar{\psi}_{\kappa_i}(x)/\|\bar{\psi}_{\kappa_i}\|_w^2;\, i\in\mathbb{N}\right\}$ where $\|f\|_{\mathfrak{w}}^2 = \langle f,\, f\rangle_{\mathfrak{w}}$ is the norm associated to the inner product (4.16) with respect to the weight function $\mathfrak{w}(x)$. Thus,

$$L^2(\mathbb{R}) = \bigoplus_{n=1}^{\infty}\mathcal{H}_n$$

where $\mathcal{H}_n$ is the space of eigenfunctions associated with the eigenvalue $\lambda_n = (\kappa_n/2)^2$ and we obtain that

$$\bar{m}^{\gamma,\mu}(x,t) = \sum_{n=1}^{\infty} c_n(t,\,\lambda_n)\,\frac{\bar{\psi}_{\kappa_n}(x)}{\|\bar{\psi}_{\kappa_n}\|_{\mathfrak{w}}^2} \tag{4.23}$$

where $\|\bar{\psi}_{\kappa_n}\|_{\mathfrak{w}} = J'_{\mu-1}(\kappa_n)/\sqrt{\gamma}$ (see, e.g. [35, p. 130]). From (4.21) we have that

$$\sum_{n=1}^{\infty}\frac{\partial^\nu}{\partial t^\nu}c_n(t,\,\lambda_n)\,\frac{\bar{\psi}_{\kappa_n}(x)}{\|\bar{\psi}_{\kappa_n}\|_{\mathfrak{w}}^2} = \sum_{n=1}^{\infty}c_n(t,\,\lambda_n)\,\mathcal{G}\,\frac{\bar{\psi}_{\kappa_n}(x)}{\|\bar{\psi}_{\kappa_n}\|_{\mathfrak{w}}^2} \tag{4.24}$$

which holds term by term. From the fact that

$$\mathcal{G}\,\bar{\psi}_{\kappa_n}(x) = -\lambda_n\,\bar{\psi}_{\kappa_n}(x)$$

where $\lambda_n = (\kappa_n/2)^2$ (see (4.6)), formula (4.24) lead to the fractional equation

$$^{c}D_t^\nu\, c_n(t,\,\lambda_n) = -\lambda_n\, c_n(t,\,\lambda_n)$$

and thus, we obtain

$$c_n(t,\,\lambda_n) = c_n\cdot E_\nu(-\lambda_n\, t^\nu) \tag{4.25}$$

(the Mittag-Leffler is an eigenfunction of the Dzhrbashyan-Caputo fractional derivative) where c_n must be determined by taking into account the initial data. In particular,

$$c_n = \langle m_0/\mathfrak{w},\, \bar{\psi}_{\kappa_n}\rangle_{\mathfrak{w}} = \int_{\Omega_1} m_0(x)\bar{\psi}_{\kappa_n}(x)\,dx.$$

Formula (4.23) solves (4.21) and we obtain

$$\bar{m}_\nu^{\gamma,\mu}(x,t) = \langle\bar{m}^{\gamma,\mu}(x,\cdot),\, l_\nu(\cdot,t)\rangle = \sum_n c_n\,E_\nu\left(-(\kappa_n/2)^2\,t^\nu\right)\,\frac{\bar{\psi}_n(x)}{\|\bar{\psi}_{\kappa_n}\|_{\mathfrak{w}}^2}. \tag{4.26}$$

We have to observe that $m_\nu^{\gamma,\mu}(x,t) = \mathfrak{w}(x)\bar{m}_\nu^{\gamma,\mu}(x,t)$ for the proof to be completed.

4.4 Proof of Theorem 3

For $\mu > 0$, $\alpha \in (0,1)$

$$\exists \mathbb{S} \subset \mathbb{H}_0^\infty \text{ s.t. } (\mathcal{T} I_{0-}^{1-\alpha} k_\mu^1)(\eta) = 0, \quad \eta \in \mathbb{S} \tag{4.27}$$

(see the Appendix A) where $g_\mu^\gamma(x,t) = \mathfrak{w}(x)\, k_\mu^\gamma(x,t)$ and $k_\mu^\gamma(x,t) = |\gamma|/\Gamma(\mu)\exp(-(x/t)^\gamma)/t^{\gamma\mu}$. Indeed, being

$$(I_{0-}^{1-\alpha} f)(x) = \frac{1}{\Gamma(\alpha)} \int_x^\infty (s-x)^{\alpha-1} f(s)\, ds, \quad x > 0$$

we obtain

$$(I_{0-}^{1-\alpha} k_\mu^1(\cdot,t))(x) = t^{\alpha-1}\, k_\mu^1(x,t)$$

and (4.27) immediately follows. We restrict ourselves to the case $\nu \in (0,1)$. From the formula (A.9), we obtain

$$\begin{aligned}
&\int_0^\infty x^{\eta-1} \frac{\partial^\nu}{\partial(-x)^\nu}\left(x^{\mu-1+\nu} \frac{\partial^\nu}{\partial(-x)^\nu}\left(x^{1-\mu}\, g_\mu^1(x,t)\right)\right) dx \\
=&\frac{\Gamma(\eta)}{\Gamma(\eta-\nu)} \int_0^\infty x^{\eta-\nu-1}\left(x^{\mu-1+\nu} \frac{\partial^\nu}{\partial(-x)^\nu}\left(x^{1-\mu}\, g_\mu^1(x,t)\right)\right) dx \\
=&\frac{\Gamma(\eta)}{\Gamma(\eta-\nu)} \int_0^\infty x^{(\eta+\mu-1)-1} \frac{\partial^\nu}{\partial(-x)^\nu}\left(x^{1-\mu}\, g_\mu^1(x,t)\right) dx \\
=&\frac{\Gamma(\eta)}{\Gamma(\eta-\nu)} \frac{\Gamma(\eta+\mu-1)}{\Gamma(\eta+\mu-1-\nu)} \int_0^\infty x^{(\eta+\mu-1-\nu)-1}\left(x^{1-\mu}\, g_\mu^1(x,t)\right) dx \\
=&\frac{\Gamma(\eta)}{\Gamma(\eta-\nu)} \frac{\Gamma(\eta+\mu-1)}{\Gamma(\eta+\mu-1-\nu)} \int_0^\infty x^{(\eta-\nu)-1} g_\mu^1(x,t)\, dx \\
=&\frac{\Gamma(\eta)}{\Gamma(\eta-\nu)} \frac{\Gamma(\eta+\mu-1)}{\Gamma(\eta+\mu-1-\nu)} \mathcal{M}[g_\mu^1(\cdot,t)](\eta-\nu).
\end{aligned}$$

The x-Mellin transform of both members of (3.20) writes

$$\frac{\partial^\nu}{\partial t^\nu} \mathcal{M}[g_\mu^1(\cdot,t)](\eta) = \frac{\Gamma(\eta)}{\Gamma(\eta-\nu)} \frac{\Gamma(\eta+\mu-1)}{\Gamma(\eta+\mu-1-\nu)} \mathcal{M}[g_\mu^1(\cdot,t)](\eta-\nu)$$

where $\mathcal{M}[g_\mu^1(\cdot,t)](\eta) = t^{\eta-1}\Gamma(\eta+\mu-1)/\Gamma(\mu)$, $\eta \in \mathbb{H}_{1-\mu}^\infty$. Thus, we have that

$$\frac{\partial^\nu}{\partial t^\nu} \mathcal{M}[g_\mu^1(\cdot,t)](\eta) = \frac{\Gamma(\eta)}{\Gamma(\eta-\nu)} \frac{\Gamma(\eta+\mu-1)}{\Gamma(\mu)} t^{\eta-1-\nu}$$

because of the fact that $\frac{\partial^\beta}{\partial t^\beta} t^{\beta-1} = \Gamma(\beta)/\Gamma(\beta-\alpha) t^{\beta-\alpha-1}$ (see e.g. [55, Property 2.5]) which concludes the proof.

4.5 Proof of Theorem 4

We proceed as follows: first of all we find out the Mellin transform of the fractional operator acting on space

$$\mathcal{A} f(x,t) = -\frac{\partial^\nu}{\partial x^\nu}\left(x^{\mu-1+\nu} \frac{\partial^\nu}{\partial(-x)^\nu}\left(x^{1-\mu}\, f(x,t)\right)\right), \quad \nu \in (0,1) \tag{4.28}$$

for a well-defined function $f \in \mathbb{M}_a^\infty$, $a \in \mathbb{R}$ (and for which (4.27) holds, that is $(\mathcal{T} I_{0\pm}^{1-\alpha} f)(\eta) = 0$) and second of all we prove *ii*) by exploiting the Mellin technique and then *i*) as a particular case of *ii*). We also consider $f \in D(\mathcal{G}^*)$. Let us write

$$\Phi_\nu(\eta) = \frac{\Gamma(1-\eta+\nu)}{\Gamma(1-\eta)} \quad \text{and} \quad \Psi_\nu(\eta) = \frac{\Gamma(\eta+\mu-1)}{\Gamma(\eta+\mu-1-\nu)}.$$

From (A.10) we have that

$$\begin{aligned}\int_0^\infty x^{\eta-1} \mathcal{A} f(x,t) dx =& - \Phi_\nu(\eta) \int_0^\infty x^{\eta-\nu-1} x^{\mu-1+\nu} \frac{\partial^\nu}{\partial(-x)^\nu}\Big(x^{1-\mu} f(x,t)\Big) dx \\ =& - \Phi_\nu(\eta) \int_0^\infty x^{\eta+\mu-1} \frac{\partial^\nu}{\partial(-x)^\nu}\Big(x^{1-\mu} f(x,t)\Big) dx.\end{aligned}$$

From (A.9) we obtain

$$\begin{aligned}\int_0^\infty x^{\eta-1} \mathcal{A} f(x,t) dx =& - \Phi_\nu(\eta)\Psi_\nu(\eta) \int_0^\infty x^{\eta+\mu-\nu-1} x^{1-\mu} f(x,t) dx \\ =& - \Phi_\nu(\eta)\Psi_\nu(\eta) \mathcal{M}[f(\cdot,t)](\eta-\nu).\end{aligned}$$

Thus, by collecting all pieces together we have that

$$\mathcal{M}[\mathcal{A} f(\cdot,t)](\eta) = -\frac{\Gamma(1-\eta+\nu)}{\Gamma(1-\eta)} \frac{\Gamma(\eta+\mu-1)}{\Gamma(\eta+\mu-1-\nu)} \mathcal{M}[f(\cdot,t)](\eta-\nu). \tag{4.29}$$

Now, we consider the x-Mellin transform

$$\begin{aligned}\mathcal{M}[\mathbf{g}_\mu^{\nu,\beta}(\cdot,t)](\eta) =& \mathcal{M}[g_\mu^1(\cdot,1)](\eta) \times \mathcal{M}[\mathfrak{f}_{\nu,\beta}(\cdot,t)](\eta) \\ =& \frac{\Gamma(\eta+\mu-1)}{\Gamma(\mu)} \mathcal{M}[\mathfrak{f}_{\nu,\beta}(\cdot,t)](\eta)\end{aligned}$$

where the fact that $h_\nu \in \mathbb{F}_\nu$ and $l_\beta \in \mathbb{F}_{1/\beta}$ leads to

$$\mathcal{M}[\mathfrak{f}_{\nu,\beta}(\cdot,t)](\eta) = \mathcal{M}[h_\nu(\cdot,1)](\eta) \times \mathcal{M}[l_\beta(\cdot,t)]\left(\frac{\eta-1}{\nu}+1\right).$$

and, from the formulas (2.17) we obtain

$$\mathcal{M}[\mathbf{g}_\mu^{\nu,\beta}(\cdot,t)](\eta) = \frac{\Gamma(\eta+\mu-1)}{\Gamma(\mu)} \frac{\Gamma\left(\frac{1-\eta}{\nu}\right)}{\nu\,\Gamma(1-\eta)} \frac{\Gamma\left(\frac{\eta-1}{\nu}+1\right)}{\Gamma\left(\frac{\eta-1}{\nu}\beta+1\right)} t^{\frac{\eta-1}{\nu}\beta}, \quad \eta \in \mathbb{H}_a^1 \tag{4.30}$$

where $a = \max\{0, 1-\mu\}$, $\mu > 0$. Now, we show that

$$\frac{\partial^\beta}{\partial t^\beta} \mathcal{M}[\mathbf{g}_\mu^{\nu,\beta}(\cdot,t)](\eta) = \mathcal{M}[\mathcal{A}\, \mathbf{g}_\mu^{\nu,\beta}(\cdot,t)](\eta) \tag{4.31}$$

by taking into account the formula (4.29). The right-hand side of the formula (4.31) can be written as

$$\begin{aligned}\mathcal{M}[\mathcal{A}\, \mathbf{g}_\mu^{\nu,\beta}(\cdot,t)](\eta) =& - \frac{\Gamma(1-\eta+\nu)}{\Gamma(1-\eta)} \frac{\Gamma(\eta+\mu-1)}{\Gamma(\eta+\mu-\nu-1)} \mathcal{M}[\mathbf{g}_\mu^{\nu,\beta}(\cdot,t)](\eta-\nu) \\ =& - \frac{\Gamma(\eta+\mu-1)}{\Gamma(\mu)} \frac{\left(\frac{1-\eta}{\nu}\right)\Gamma\left(\frac{1-\eta}{\nu}\right)}{\nu\,\Gamma(1-\eta)} \frac{\Gamma\left(\frac{\eta-1}{\nu}\right)}{\Gamma\left(\frac{\eta-1}{\nu}\beta-\beta+1\right)} t^{\frac{\eta-1}{\nu}\beta-\beta}\end{aligned}$$

$$=\frac{\Gamma(\eta+\mu-1)}{\Gamma(\mu)}\frac{\Gamma\left(\frac{1-\eta}{\nu}\right)}{\nu\,\Gamma(1-\eta)}\frac{\Gamma\left(\frac{\eta-1}{\nu}+1\right)}{\Gamma\left(\frac{\eta-1}{\nu}\beta-\beta+1\right)}t^{\frac{\eta-1}{\nu}\beta-\beta}$$
$$=\mathcal{M}[\mathbf{g}_\mu^{\nu,\beta}(\cdot,1)](\eta)\frac{\Gamma\left(\frac{\eta-1}{\nu}\beta+1\right)}{\Gamma\left(\frac{\eta-1}{\nu}\beta-\beta+1\right)}t^{\frac{\eta-1}{\nu}\beta-\beta}.$$

From the fact that

$$\frac{\partial^\beta}{\partial t^\beta}\,t^{\frac{\eta-1}{\nu}\beta}=\frac{\Gamma\left(\frac{\eta-1}{\nu}\beta+1\right)}{\Gamma\left(\frac{\eta-1}{\nu}\beta-\beta+1\right)}t^{\frac{\eta-1}{\nu}\beta-\beta}$$

(see [55, Property 2.5]) formula (4.31) immediately follows and this prove ii).

For $\beta=1$, the formula (4.30) takes the form

$$\mathcal{M}[\mathfrak{g}_\mu^\nu(\cdot,t)](\eta)=\frac{\Gamma(\eta+\mu-1)}{\nu\,\Gamma(\mu)\,\Gamma(1-\eta)}\Gamma\left(\frac{1-\eta}{\nu}\right)t^{\frac{\eta-1}{\nu}},\quad \eta\in\mathbb{H}_a^1$$

where $a=\max\{0,1-\mu\}$. This is (for $\beta=1$) because of the fact that $\mathfrak{H}^\nu_{\mathfrak{L}^1_t}\overset{a.s.}{=}\mathfrak{H}^\nu_t$, $t>0$, being $\mathfrak{L}^1_t\overset{a.s.}{=}t$ the elementary subordinator, see e.g. Bertoin [8]. Thus, form (4.29), the Mellin transform of both members of (3.25) becomes

$$-\frac{\partial}{\partial t}\mathcal{M}[\mathfrak{g}_\mu^\nu(\cdot,t)](\eta)=\frac{\Gamma(1-\eta+\nu)}{\Gamma(1-\eta)}\frac{\Gamma(\eta+\mu-1)}{\Gamma(\eta+\mu-1-\nu)}\mathcal{M}[\mathfrak{g}_\mu^\nu(\cdot,t)](\eta-\nu)$$

where

$$\begin{aligned}\mathcal{M}[\mathfrak{g}_\mu^\nu(\cdot,t)](\eta-\nu)=&\frac{\Gamma(\eta+\mu-1-\nu)}{\pi\nu\,\Gamma(\mu)\,\Gamma(1-\eta+\nu)}\Gamma\left(\frac{1-\eta+\nu}{\nu}\right)t^{\frac{\eta-\nu-1}{\nu}}\\=&-\left(\frac{\eta-1}{\nu}\right)\frac{\Gamma(\eta+\mu-1-\nu)}{\pi\nu\,\Gamma(\mu)\,\Gamma(1-\eta+\nu)}\Gamma\left(\frac{1-\eta}{\nu}\right)t^{\frac{\eta-1}{\nu}-1}\\=&-\frac{\Gamma(\eta+\mu-1-\nu)}{\pi\nu\,\Gamma(\mu)\,\Gamma(1-\eta+\nu)}\Gamma\left(\frac{1-\eta}{\nu}\right)\frac{\partial}{\partial t}t^{\frac{\eta-1}{\nu}}.\end{aligned}$$

By collecting all pieces together we obtain the result claimed in i).

From (4.30) and by direct inspection of (A.12) we arrive at

$$\mathcal{M}[\mathbf{G}_\mu^{\nu,\beta}(\cdot)](\eta)=\mathcal{M}^{2,1}_{3,3}\left[\eta\left|\begin{array}{lll}(1-\frac{1}{\nu},\frac{1}{\nu}); & (1-\frac{\beta}{\nu},\frac{\beta}{\nu}); & (\mu,0)\\ (\mu-1,1); & (1-\frac{1}{\nu},\frac{1}{\nu}); & (0,1)\end{array}\right.\right]$$

where $\mathbf{G}_\mu^{\nu,\beta}(x)=\mathbf{g}_\mu^{\nu,\beta}(x,1)$. Thus,

$$\begin{aligned}\mathbf{G}_\mu^{\nu,\beta}(x)=&H^{2,1}_{3,3}\left[x\left|\begin{array}{lll}(1-\frac{1}{\nu},\frac{1}{\nu}); & (1-\frac{\beta}{\nu},\frac{\beta}{\nu}); & (\mu,0)\\ (\mu-1,1); & (1-\frac{1}{\nu},\frac{1}{\nu}); & (0,1)\end{array}\right.\right]\\=&\frac{1}{x}H^{2,1}_{3,3}\left[x\left|\begin{array}{lll}(1,\frac{1}{\nu}); & (1,\frac{\beta}{\nu}); & (\mu,0)\\ (\mu,1); & (1,\frac{1}{\nu}); & (1,1)\end{array}\right.\right]\end{aligned}$$

where we used, for $c=1$, the property of the H functions

$$H^{m,n}_{p,q}\left[x\left|\begin{array}{l}(a_i,\alpha_i)_{i=1,..,p}\\(b_j,\beta_j)_{j=1,..,q}\end{array}\right.\right]=\frac{1}{x^c}\,H^{m,n}_{p,q}\left[x\left|\begin{array}{l}(a_i+c\alpha_i,\alpha_i)_{i=1,..,p}\\(b_j+c\beta_j,\beta_j)_{j=1,..,q}\end{array}\right.\right]\tag{4.32}$$

for all $c\in\mathbb{R}$ (see Mathai and Saxena [41]). From (A.2), by observing that

$$\mathcal{M}\left[\frac{1}{t^{\beta/\nu}}\mathbf{G}_\mu^{\nu,\beta}\left(\frac{\cdot}{t^{\beta/\nu}}\right)\right](\eta)=\mathcal{M}[\mathbf{G}_\mu^{\nu,\beta}(\cdot)](\eta)t^{\frac{\eta-1}{\nu}\beta}$$

we obtain the claimed result.

4.6 Proof of Theorem 5

Hereafter, we extend the result given in [16] (Lemmas 3 and 4) and show how the Mellin convolution turns out to be useful in order to explicitly write the distributions of both stable subordinators and their inverse processes. Let us consider the time-stretching functions $\psi_m(s) = m\, s^{1/m}$, $s \in (0,\infty)$, $m \in \mathbb{N}$ and φ_m such that $\psi_m = \varphi_m^{-1}$ (the inverse function of φ_m).

Lemma 3. *[16, Lemma 2] The Mellin convolution $e_{\boldsymbol{\mu}}^{\star n}(x, \varphi_{n+1}(t))$ where $\mu_j = j\,\nu$, for $j = 1, 2, \ldots, n$ is the density law of a ν-stable subordinator $\{H_t^{(\nu)},\, t > 0\}$ with $\nu = 1/(n+1)$, $n \in \mathbb{N}$. Thus, we have*

$$h_\nu(x,t) = e_{\boldsymbol{\mu}}^{\star n}(x, \varphi_{n+1}(t)), \quad x, t > 0.$$

We recall that $e_\mu = g_\mu^{-1}$ is the 1-dimensional law of E_μ.

Lemma 4. *[16, Lemma 3] The Mellin convolution $g_{\boldsymbol{\mu}}^{(n+1),\star n}(x, \psi_{n+1}(t))$ where $\mu_j = j\,\nu$, $j = 1, 2, \ldots, n$ and $\nu = 1/(n+1)$, $n \in \mathbb{N}$, is the density law of a ν-inverse process $\{L_t^{(\nu)},\, t > 0\}$. Thus, we have*

$$l_\nu(x,t) = g_{\boldsymbol{\mu}}^{(n+1),\star n}(x, \psi_{n+1}(t)), \quad x, t > 0.$$

We observe that g_μ^{n+1} is the 1-dimensional law of $G_\mu^{\frac{1}{n+1}}(t^{n+1}) = G_\mu^{\nu}(t^{1/\nu})$.

The following facts will be useful later on.

Lemma 5. *For $g_\mu^\gamma = g_\mu^\gamma(x,t)$, $x \in \mathbb{R}_+$, $t > 0$, $\mu > 0$ the following hold:*

i) $\star$-commutativity: $g_{\mu_1}^{\gamma_1} \star g_{\mu_2}^{\gamma_2} = g_{\mu_2}^{\gamma_2} \star g_{\mu_1}^{\gamma_1}$ for all $\gamma_1, \gamma_2 \neq 0$.

ii) $*$-commutativity: $g_{\mu_1}^{\gamma_1} * g_{\mu_2}^{\gamma_2} = g_{\mu_2}^{\gamma_2} * g_{\mu_1}^{\gamma_1}$ for all $\gamma_1, \gamma_2 \neq 0$. Furthermore, for $\gamma = 1, 2$,

$$g_{\mu_1}^{\gamma} * g_{\mu_2}^{\gamma} = g_{\mu_1+\mu_2}^{\gamma}.$$

iii) $(\star,*)$-distributivity: when $\star$- and $*$- commutativity hold, we have

$$g_{\mu_1}^{\gamma} \star (g_{\mu_2}^{\gamma} * g_{\mu_3}^{\gamma}) = (g_{\mu_1}^{\gamma} \star g_{\mu_2}^{\gamma}) * (g_{\mu_1}^{\gamma} \star g_{\mu_3}^{\gamma}).$$

iv) for $\mu_1, \mu_2, c \in \mathbb{N}$

$$g_{\mu_1 \cdot \mu_2}^{1} = \circledast_{j_1=1}^{\mu_1} g_{\mu_2}^{1} = \circledast_{j_2=1}^{\mu_2} g_{\mu_1}^{1} \quad \text{and} \quad \circledast_{j_1=1}^{\mu_1 \pm c}\, g_{\mu_2}^{1} = g_{\mu_1 \cdot \mu_2 \cdot c^{\pm 1}}^{1} \tag{4.33}$$

where $\circledast_{j=1}^{n} f = f_1 * f_2 *, \ldots, * f_n$.

Proof. The point $i)$ comes directly from the formula (3.34) and the fact that $g_{\mu_j}^{\gamma_j} \in \mathbb{F}_1$, $\forall \gamma_j \neq 0$, and $\mu_j > 0$, $j = 1, 2$. We show that $ii)$ holds. For $\gamma = 1$, $\forall\, t > 0$, g_μ^γ is the gamma density with Laplace transform $\mathcal{L}[g_\mu^1(\cdot, t)](\lambda) = 1/(1 + \lambda t)^\mu$ and the statement follows easily. This is a well-known result. The case $\gamma = 2$ is considered in Shiga and Watanabe [57] being g_μ^2 the semigroup for a Bessel process $R_{2\mu} = G_{2\mu}^{1/2}$ where G_μ satisfies the stochastic equation (3.5). The result in $iii)$ can be obtained by considering, $\forall\, t > 0$, the independent r.v.'s $G_{\mu_j}^{\gamma_j}(t)$, $j = 1, 2, 3$ with densities

$g^{\gamma_j}_{\mu_j} = g^{\gamma_j}_{\mu_j}(x,t)$, $j = 1,2,3$, $x \in \mathbb{R}_+$. From the fact that $g^{\gamma_j}_{\mu_j} \in \mathbb{F}_1$, $j = 1,2,3$ we have that $G^{\gamma_1}_{\mu_1}(G^{\gamma_2}_{\mu_2}(t)) \overset{law}{=} G^{\gamma_2}_{\mu_2}(G^{\gamma_1}_{\mu_1}(t))$, that is, $\forall\, j$, $g^{\gamma_j}_{\mu_j}$ are commutative under $\star$. For this reason and the $*$-commutativity, $\forall\, t > 0$, we can write

$$\begin{aligned} G^1_{\mu_1}(G^1_{\mu_2}(t) + G^1_{\mu_3}(t)) =& G^1_{\mu_1}(G^1_{\mu_2+\mu_3}(t)) = G^1_{\mu_2+\mu_3}(G^1_{\mu_1}(t)) \\ =& G^1_{\mu_2}(G^1_{\mu_1}(t)) + G^1_{\mu_3}(G^1_{\mu_1}(t)). \end{aligned}$$

In the last calculation we have used the fact that

$$E\Big[\exp\left(-\lambda[G_{\mu_1}(s) + G_{\mu_2}(s)]\right)\Big| s = X_t\Big] = E\Big[\exp\left(-\lambda G_{\mu_1+\mu_2}(s)\right)\Big| s = X_t\Big].$$

The same result can be achieved for $\gamma = 2$. In order to prove $iv)$ we proceed as follows: first of all we observe that $ii)$ implies $g^1_{\mu_1\cdot\mu_2} = \circledast^{\mu_1}_{j_1=1} g^1_{\mu_2} = \circledast^{\mu_2}_{j_2=1} g^1_{\mu_1}$. Second of all we consider that

$$\circledast^{\mu_1+c}_{j_1=1} g^1_{\mu_2} = \circledast^{c}_{j_c=1} \circledast^{\mu_1}_{j_1=1} g^1_{\mu_2} = g^1_{\mu_1\cdot\mu_2\cdot c}$$

whereas

$$\circledast^{c}_{j_c=1} \circledast^{\mu_1-c}_{j_1=1} g^1_{\mu_2} = \circledast^{c}_{j_c=1} g^1_{\frac{\mu_1\cdot\mu_2}{c}} = g^1_{\mu_1\cdot\mu_2}$$

and this concludes the proof. □

Proposition 1. *The following holds true*

$$g^{\gamma,\star n}_{\boldsymbol{\mu}}(x,t) = \tilde{g}^{\gamma}_{\mu_j} \circ g^{1,\star(n-1)}_{\boldsymbol{\mu}\setminus\{\mu_j\}}(x,t^{\gamma}), \quad \forall \mu_j \in \boldsymbol{\mu},\ j = 1,2,\dots,n \tag{4.34}$$

where

$$f_1 \circ f_2(x,t) = \int_0^{\infty} f_1(x,s)\, f_2(s,t)\, ds$$

for $f_j : [0,+\infty) \mapsto [0,+\infty)$, $j = 1,2$.

Proof. Fix $n = 3$. $\forall\, t > 0$, it is enough to consider the r.v.'s G^{γ}_{μ}, $\tilde{G}^{\gamma}_{\mu}$ and their density laws g^{γ}_{μ}, $\tilde{g}^{\gamma}_{\mu}$ where $\tilde{g}^{\gamma}_{\mu}(x,t) = g^{\gamma}_{\mu}(x,t^{1/\gamma})$ or equivalently $G^{\gamma}_{\mu}(t) \overset{law}{=} \tilde{G}^{\gamma}_{\mu}(t^{\gamma})$. In this setting, we have that $X(t) = G^{\gamma}_{\mu_1}(G^{\gamma}_{\mu_2}(G^{\gamma}_{\mu_3}(t)))$ can be written as $X(t) \overset{law}{=} \tilde{G}^{\gamma}_{\mu_1}(G^1_{\mu_2}(G^1_{\mu_3}(t^{\gamma})))$ thank to the fact that $\left(G^{\gamma}_{\mu}\right)^{\gamma} \overset{law}{=} G^1_{\mu}$. Thus, we can write

$$g^{\gamma,\star 3}_{\boldsymbol{\mu}}(x,t) = \tilde{g}^{\gamma}_{\mu_1} \circ g^{1,\star 2}_{(\mu_2,\mu_3)}(x,t^{\gamma}) = \tilde{g}^{\gamma}_{\mu_1} \circ (g^1_{\mu_2} \star g^1_{\mu_3})(x,t^{\gamma}).$$

Thanks to the $\star$-commutativity we have that $g^1_{\mu_2} \star g^1_{\mu_3} = g^1_{\mu_3} \star g^1_{\mu_2}$ and also that

$$g^{\gamma,\star 3}_{\boldsymbol{\mu}}(x,t) = \tilde{g}^{\gamma}_{\mu_2} \circ g^{1,\star 2}_{(\mu_1,\mu_3)}(x,t^{\gamma}) = \tilde{g}^{\gamma}_{\mu_2} \circ (g^1_{\mu_1} \star g^1_{\mu_3})(x,t^{\gamma}).$$

By considering n processes, the formula (4.34) immediately appears. □

Remark 9. For $\nu = 1/5$, we have that

$$l_{1/5}(x,t) = g^{5,\star 4}_{\boldsymbol{\mu}_1}(x, 5t^{1/5}), \quad \boldsymbol{\mu}_1 = (1/5, 2/5, 3/5, 4/5) \tag{4.35}$$

and, from the Proposition 1 and the Lemma 5,

$$\begin{aligned} g^{5,\star 4}_{(1/5,2/5,3/5,4/5)} =& \tilde{g}^5_{1/5} \circ g^{1,\star 3}_{(4/5,3/5,2/5)} \\ =& \tilde{g}^5_{1/5} \circ \left[(g^1_{1/5} * g^1_{1/5} * g^1_{1/5} * g^1_{1/5}) \star (g^1_{1/5} * g^1_{1/5} * g^1_{1/5}) \star g^1_{2/5}\right] \end{aligned}$$

$$
\begin{aligned}
=&\tilde{g}^5_{1/5} \circ \left[(g^1_{1/5} * g^1_{1/5} * g^1_{1/5} * g^1_{1/5}) \star (g^{1,\star 2}_{(2/5,1/5)} * g^{1,\star 2}_{(2/5,1/5)} * g^{1,\star 2}_{(2/5,1/5)}) \right] \\
=&\tilde{g}^5_{1/5} \circ \left[\divideontimes^{12}_{k=1} \left(g^{1,\star 3}_{(2/5,1/5,1/5)} \right)_k \right] \\
=&\tilde{g}^5_{1/5} \circ g^{1,\star 3}_{24/5,1/5,1/5}(x, 5^5 t) = g^{5,\star 4}_{(24/5,1/5,1/5,1/5)}
\end{aligned}
$$

where

$$
\boldsymbol{\mu}_2 = (24/5, 1/5, 1/5, 1/5)
$$

and $\boldsymbol{\mu}_1, \boldsymbol{\mu}_2 \in \mathscr{P}^4_5\,(24)$. Finally, we obtain

$$
g^{5,\star 4}_{(24/5,1/5,1/5,1/5)}(x,t) = g^{5,\star 2}_{(24/5,1/5)} \star g^{5,\star 2}_{(1/5,1/5)}(x,t).
$$

From (3.31) the corresponding integral representation emerges.

Theorem 6. *Fix $\boldsymbol{\mu} \in \mathscr{P}^n_\kappa\,(\varrho)$. Then $g^{1,\star n}_{\boldsymbol{\mu}} = g^{1,\star n}_{\boldsymbol{\vartheta}}$ for all $\boldsymbol{\vartheta} \in \mathscr{P}^n_\kappa\,(\varrho)$.*

Proof. Fix $\kappa, \varrho \in \mathbb{N}$. We have $g^{1,\star n}_{\boldsymbol{\mu}} = g^1_{\mu_1} \star \ldots \star g^1_{\mu_n}$, $\boldsymbol{\mu} = (\mu_1, \ldots \mu_n) \in \mathscr{P}^n_\kappa(\varrho)$. From (3.38) we can write $\boldsymbol{\mu} = \frac{1}{\kappa}(\tilde{\mu}_1, \ldots, \tilde{\mu}_n)$. Let us first consider $n = 2$. We recall that $g^{1,\star 2}_{(\mu_1,\mu_2)} = g^{1,\star 2}_{(\mu_2,\mu_1)}$ from the $\star$-commutativity. Thus, form the properties $i)$ and $ii)$ of the Lemma 5, we have that

$$
\begin{aligned}
g^{1,\star 2}_{(\frac{1}{\kappa}\tilde{\mu}_1, \frac{1}{\kappa}\tilde{\mu}_2)} =& \divideontimes^{\tilde{\mu}_1}_{j_1=1} g^{1,\star 2}_{(\frac{1}{\kappa}, \frac{1}{\kappa}\tilde{\mu}_2)} = \divideontimes^{\tilde{\mu}_1}_{j_1=1} g^{1,\star 2}_{(\frac{1}{\kappa}\tilde{\mu}_2, \frac{1}{\kappa})} \\
=& \divideontimes^{\tilde{\mu}_1}_{j_1=1} \divideontimes^{\tilde{\mu}_2}_{j_2=1} g^{1,\star 2}_{(\frac{1}{\kappa}, \frac{1}{\kappa})} = \divideontimes^{\tilde{\mu}_1}_{j_1=1} \divideontimes^{\tilde{\mu}_2}_{j_2=1} g^{1,\star 2}_{\frac{1}{\kappa}(1,1)}.
\end{aligned}
$$

For $n \in \mathbb{N}$, $\boldsymbol{\vartheta}_0 = \frac{1}{\kappa}(1, \ldots, 1) \in \mathbb{R}^n_+$, we can write

$$
g^{1,\star n}_{\boldsymbol{\mu}} = g^{1,\star n}_{\frac{1}{\kappa}(\tilde{\mu}_1, \ldots, \tilde{\mu}_n)} = \divideontimes^{\tilde{\mu}_1}_{j_1=1} \cdots \divideontimes^{\tilde{\mu}_n}_{j_n=1} g^{1,\star n}_{\boldsymbol{\vartheta}_0}.
$$

We shall refer to $\boldsymbol{\vartheta}_0$ as the 0-configuration. We first observe that

$$
g^{1,\star n}_{\boldsymbol{\mu}} = \divideontimes^{\tilde{\mu}_1}_{j_1=1} \cdots \divideontimes^{\tilde{\mu}_n}_{j_n=1} g^{1,\star n}_{\boldsymbol{\vartheta}_0} = g^{1,\star n}_{\frac{1}{\kappa}(\varrho,1,\ldots,1)}, \quad \varrho = \prod_{j=1}^n \tilde{\mu}_j,
$$

or equivalently

$$
g^{1,\star n}_{\boldsymbol{\mu}} = \divideontimes^{\tilde{\mu}_1}_{j_1=1} \cdots \divideontimes^{\tilde{\mu}_n}_{j_n=1} g^{1,\star n}_{\boldsymbol{\vartheta}_0} = \divideontimes^{\tilde{\mu}_i}_{j_i=1} g^{1,\star n}_{\frac{1}{\kappa}(\varrho/\tilde{\mu}_i,1,\ldots,1)} = g^{1,\star n}_{\frac{1}{\kappa}(\varrho_i,\tilde{\mu}_i,1,\ldots,1)}
$$

$\varrho_i = \varrho/\tilde{\mu}_i$, for all $i = 1, 2, \ldots, n$. The last identity comes from the $\star$-commutativity. By exploiting the $*$-commutativity and the $\star$-commutativity we have that $g^{1,\star n}_{\boldsymbol{\mu}} = g^{1,\star n}_{\boldsymbol{\theta}}$ for all $\boldsymbol{\theta}$ such that

$$
\mathbb{R}^n_+ \ni \boldsymbol{\theta} = \frac{1}{\kappa}(\varrho_{\mathbf{i}}, \tilde{\mu}_{\mathbf{i}}, \mathbf{1}), \quad \varrho_{\mathbf{i}} = \varrho / \prod_{s_j=1}^{|\mathbf{i}|} \tilde{\mu}_{s_j}
$$

where $dim(\mathbf{1}) = n - |\mathbf{i}| - 1$, $\tilde{\mu}_{\mathbf{i}} = (\tilde{\mu}_{s_1}, \ldots, \tilde{\mu}_{s_{|\mathbf{i}|}}) \in \mathbb{N}^{|\mathbf{i}|}_+$, $s_j \in \mathbf{i}$, $j = 1, 2, \ldots |\mathbf{i}|$ and $|\mathbf{i}| < n$ is the cardinality of $\mathbf{i}$. A further configuration is given by $\bar{\boldsymbol{\theta}} = (\varrho_{\mathbf{i}}, \tilde{\mu}_{\mathbf{i}})/\kappa$ where $|\mathbf{i}| = n - 1$. In this case, $\bar{\boldsymbol{\theta}} \in \mathscr{P}^n_\kappa(\varrho)$ is obtainable by $n!$ permutation of the elements of $\boldsymbol{\mu}$. In a more general setting, for $\bar{\alpha} = (\alpha_1 \cdot \beta, \alpha_2, \ldots, \alpha_n) \in \mathbb{N}^n$, $\beta \in \mathbb{N}$, $c \in \mathbb{N}$ the following rules hold

$$
g^{1,\star n}_{\bar{\alpha}} = \divideontimes^{\beta}_{j=1} g^{1,\star n}_{(\alpha_1, \alpha_2, \ldots, \alpha_n)} \tag{4.36}
$$

(see iv), Lemma 5) and, for $c \geq 1$,

$$\circledast_{j=1}^{\beta\pm c}\, g^{1,\star n}_{(\alpha_1,\alpha_2,\ldots,\alpha_n)} = \circledast_{j=1}^{\beta\pm c} g^{1,\star n}_{(\alpha_{\sigma_1},\alpha_{\sigma_2},\ldots,\alpha_{\sigma_n})} = g^{1,\star n}_{(\alpha_{\sigma_1}\cdot\beta\cdot c^{\pm 1},\alpha_{\sigma_2},\ldots,\alpha_{\sigma_n})} \tag{4.37}$$

for all permutation of $\{\sigma_j\}$, $j = 1, 2, \ldots, n$. We recall that

$$\circledast_{j=1}^{\beta+c} g_\alpha = \circledast_{j_1=1}^{c} \circledast_{j_2=1}^{\beta}\, g_\alpha = g_{\alpha\beta c},$$

for $\alpha, \beta, c \in \mathbb{N}$. By making use of the properties i), ii) and iii) of the Lemma 5 we can obtain all possible configurations of $\boldsymbol{\vartheta} \in \mathbb{R}^n_+$ starting from the 0-configuration $\boldsymbol{\vartheta}_0$. All different configurations of $\boldsymbol{\vartheta}$ are included in $\mathscr{P}^n_\kappa(\varrho)$. From (4.37), for all $\boldsymbol{\vartheta} = (\vartheta_1, \ldots, \vartheta_n) \in \mathscr{P}^n_\kappa(\varrho)$ we have that $\prod_{j=1}^n \vartheta_j = \varrho$. This concludes the proof. □

By collecting all pieces together we obtain the claimed result.

A Fox functions and Mellin transform

The Mellin transform of $f \in \mathbb{M}^b_a$ (see Definition 1) is defined as

$$\mathcal{M}[f(\cdot)](\eta) = \int_0^\infty x^{\eta-1} f(x)dx, \quad \eta \in \mathbb{H}^b_a. \tag{A.1}$$

Let us point out some useful operational rules that will be useful throughout the paper: for some $-\infty < a < b < \infty$ and $\mathfrak{b} > 0$, $f, f_1, f_2 \in \mathbb{M}^b_a$:

$$\int_0^\infty x^{\eta-1} f(\mathfrak{b}x)dx = \mathfrak{b}^{-\eta}\mathcal{M}\left[f(\cdot)\right](\eta), \tag{A.2}$$

$$\mathcal{M}\left[x^{\mathfrak{b}} f(\cdot)\right](\eta) = \mathcal{M}\left[f(\cdot)\right](\eta+\mathfrak{b}), \tag{A.3}$$

$$\mathcal{M}\left[\int_0^\infty f_1\left(\frac{\cdot}{s}\right) f_2(s)\frac{ds}{s}\right](\eta) = \mathcal{M}\left[f_1(\cdot)\right](\eta) \times \mathcal{M}\left[f_2(\cdot)\right](\eta), \tag{A.4}$$

$$\mathcal{M}[I(\cdot)](\eta) = \eta^{-1}\mathcal{M}[f(\cdot)](\eta+1), \tag{A.5}$$

where

$$I(x) = \int_x^\infty f(s)ds, \quad x > 0, \tag{A.6}$$

see e.g. Glaeske et al. [23]. The formula (A.4) is the well-known Mellin convolution formula which turns out to be useful in the study of the product of random variables.

We say that $f \in \widetilde{\mathbb{M}}_n$ if $f \in \mathbb{M}^b_a$ and is a rapidly decreasing function such that

$$\exists \mathfrak{a} \in \mathbb{R}\ s.t. \lim_{x\to+\infty} x^{\mathfrak{a}-k-1}\frac{d^k f}{dx^k}(x) = 0, \quad k = 0, 1, \ldots, n, \quad n \in \mathbb{N},\ x \in \mathbb{R}_+$$

and

$$\exists \mathfrak{b} \in \mathbb{R}\ s.t. \lim_{x\to 0^+} x^{\mathfrak{b}-k-1}\frac{d^k f}{dx^k}(x) = 0, \quad k = 0, 1, \ldots, n, \quad n \in \mathbb{N},\ x \in \mathbb{R}_+.$$

For $f \in \widetilde{\mathbb{M}}_{n-1}$ and $n \in \mathbb{N}$ we have that

$$\mathcal{M}\left[\frac{d^n f}{dx^n}(\cdot)\right](\eta) = (-1)^n \frac{\Gamma(\eta)}{\Gamma(\eta-n)}\mathcal{M}\left[f(\cdot)\right](\eta-n) \tag{A.7}$$

$$
= \frac{\Gamma(1+n-\eta)}{\Gamma(1-\eta)} \mathcal{M}\left[f(\cdot)\right](\eta - n). \tag{A.8}
$$

As a generalized version of the integer derivatives (A.7) and (A.8) we introduce the Mellin transform of fractional derivatives (2.7) and (2.8) (see Kilbas et al. [29]; Samko et al. [55] for details). For a given $f \in \mathbb{M}_a^b$ and $0 < \alpha < 1$, if $\Re\{\eta\} > 0$,

$$
\mathcal{M}\left[\frac{d^\alpha f}{d(-x)^\alpha}(\cdot)\right](\eta) = \frac{\Gamma(\eta)}{\Gamma(\eta-\alpha)} \mathcal{M}[f(\cdot)](\eta-\alpha) + (\mathcal{T} I_{0-}^{1-\alpha} f)(\eta) \tag{A.9}
$$

whereas, if $\Re\{\eta\} < \alpha + 1$,

$$
\mathcal{M}\left[\frac{\partial^\alpha f}{\partial x^\alpha}(\cdot)\right](\eta) = \frac{\Gamma(1+\alpha-\eta)}{\Gamma(1-\eta)} \mathcal{M}\left[f(\cdot)\right](\eta-\alpha) + (\mathcal{T} I_{0+}^{1-\alpha} f)(\eta) \tag{A.10}
$$

where

$$
(\mathcal{T} I_{0\pm}^{1-\alpha} f)(\eta) = \frac{\Gamma(\alpha-\eta)}{\Gamma(1-\eta)} \left[x^{\eta-1} (I_{0\pm}^{1-\alpha} f)(x)\right]_{x=0}^{x=\infty}
$$

and

$$
(I_{0-}^{1-\alpha} f)(x) = \frac{1}{\Gamma(\alpha)} \int_x^\infty (s-x)^{\alpha-1} f(s)\, ds, \quad x > 0,
$$

$$
(I_{0+}^{1-\alpha} f)(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-s)^{\alpha-1} f(s)\, ds, \quad x > 0
$$

are the right and left fractional integrals.

The Fox functions, also referred to as Fox's H-functions, H-functions, generalized Mellin-Barnes functions, or generalized Meijer's G-functions, were introduced by Fox [21] in 1996. Here, the Fox's H-functions will be recalled as the class of functions uniquely identified by their Mellin transforms. A function $f \in \mathbb{M}_a^b$ can be written in terms of H-functions by observing that

$$
\int_0^\infty x^\eta H_{p,q}^{m,n}\left[x \left| \begin{array}{l} (a_i,\alpha_i)_{i=1,..,p} \\ (b_j,\beta_j)_{j=1,..,q} \end{array} \right.\right] \frac{dx}{x} = \mathcal{M}_{p,q}^{m,n}\left[\eta \left| \begin{array}{l} (a_i,\alpha_i)_{i=1,..,p} \\ (b_j,\beta_j)_{j=1,..,q} \end{array} \right.\right] \tag{A.11}
$$

with $\eta \in \mathbb{H}_a^b$ where

$$
\mathcal{M}_{p,q}^{m,n}\left[\eta \left| \begin{array}{l} (a_i,\alpha_i)_{i=1,..,p} \\ (b_j,\beta_j)_{j=1,..,q} \end{array} \right.\right] = \frac{\prod_{j=1}^m \Gamma(b_j+\eta\beta_j) \prod_{i=1}^n \Gamma(1-a_i-\eta\alpha_i)}{\prod_{j=m+1}^q \Gamma(1-b_j-\eta\beta_j) \prod_{i=n+1}^p \Gamma(a_i+\eta\alpha_i)}. \tag{A.12}
$$

Thus, according to a standard notation, the Fox H-function can be defined as follows

$$
H_{p,q}^{m,n}\left[x \left| \begin{array}{l} (a_i,\alpha_i)_{i=1,..,p} \\ (b_j,\beta_j)_{j=1,..,q} \end{array} \right.\right] = \frac{1}{2\pi i} \int_{\mathcal{P}(\mathbb{H}_a^b)} \mathcal{M}_{p,q}^{m,n}(\eta) x^{-\eta} d\eta \tag{A.13}
$$

where $\mathcal{P}(\mathbb{H}_a^b)$ is a suitable path in the complex plane $\mathbb{C}$ depending on the fundamental strip $(\mathbb{H}_a^b)$ such that the integral (A.11) converges. For an extensive discussion on this functions see Fox [21]; Kilbas et al. [29]; Mathai and Saxena [41].

References

[1] B. Baeumer and M. Meerschaert. Stochastic solutions for fractional Cauchy problems. *Fract. Calc. Appl. Anal.*, 4(4):481 – 500, 2001.

[2] B. Baeumer, M. M. Meerschaert, and E. Nane. Brownian subordinators and fractional Cauchy problems. *Trans. Amer. Math. Soc.*, 361:3915 – 3930, 2009.

[3] A. Balakrishnan. Fractional powers of closed operators and semigroups generated by them. *Pacific J. Math.*, 10:419 – 437, 1960.

[4] E. G. Bazhlekova. Subordination principle for fractional evolution equations. *Frac. Calc. Appl. Anal.*, 3:213 – 230, 2000.

[5] L. Beghin and E. Orsingher. Iterated elastic Brownian motions and fractional diffusion equations. *Stoch. Proc. Appl.*, 119(6):1975 – 2003, 2009.

[6] L. Beghin and E. Orsingher. The telegraph process stopped at stable-distributed times and its connection with the fractional telegraph equation. *Fract. Calc. Appl. Anal.*, 6:187 – 204, 2003.

[7] D. Benson, S. Wheatcraft, and M. Meerschaert. The fractional-order governing equation of Lévy Motion. *Water Resources Res.*, 36:1413 – 1424, 2000.

[8] J. Bertoin. *Lévy Processes.* Cambridge University Press, 1996.

[9] B. Bibby, I. Skovgaard, and M. Sorensen. Diffusion-type models with given marginal distribution and autocorrelation function. *Bernoulli*, 11(2):191 – 220, 2005.

[10] S. Bochner. Über Sturm-Liouvillesche Polynomsysteme. *Math. Zeit.*, 29:730 – 736, 1929.

[11] S. Bochner. Diffusion equation and stochastic processes. *Proc. Nat. Acad. Sciences, U.S.A.*, 35:368 – 370, 1949.

[12] L. Chaumont and M. Yor. *Exercises in probability. A guided tour from measure theory to random processes, via conditioning.* Cambridge Series in Statistical and Probabilistic Mathematics, 13. Cambridge University Press, 2003.

[13] A. Chaves. A fractional diffusion equation to describe Lévy flights. *Phys. Lett. A*, 239:13 – 16, 1998.

[14] R. Courant and D. Hilbert. *Methods of Mathematical Physics*, volume I. John Wiley & Sons, New York, 1989.

[15] R. D. DeBlassie. Higher order PDEs and symmetric stable process. *Probab. Theory Rel. Fields*, 129:495 – 536, 2004.

[16] M. D'Ovidio. Explicit solutions to fractional diffusion equations via generalized gamma convolution. *Elect. Comm. in Probab.*, 15:457 – 474, 2010.

[17] M. D'Ovidio. On the fractional counterpart of the higher-order equations. *Statistics & Probability Letters*, 81:1929 – 1939, 2011.

[18] M. D'Ovidio and E. Orsingher. Bessel processes and hyperbolic Brownian motions stopped at different random times. *Stochastic Processes and their Applications*, 121:441 – 465, 2011.

[19] D. Dufresne. The distribution of a perpetuity, with applications to risk theory and pension funding. *Scand. Actuar. J.*, 39:39 – 79, 1990.

[20] W. Feller. On a generalization of Marcel Riesz' potentials and the semigroups generated by them. *Communications du seminaire mathematique de universite de Lund, tome suppli-mentaire*, 1952. dedie' a Marcel Riesz.

[21] C. Fox. The G and H functions as symmetrical Fourier kernels. *Trans. Amer. Math. Soc.*, 98:395 – 429, 1961.

[22] M. Giona and H. Roman. Fractional diffusion equation on fractals: one-dimensional case and asymptotic behavior. *J. Phys. A*, 25:2093 – 2105, 1992.

[23] H. Glaeske, A. Prudnikov, and K. Skòrnik. *Operational Calculus and Related Topics*. Chapman & Hall/CRC. Taylor & Francis Group, 2006.

[24] R. Gorenflo and F. Mainardi. Fractional calculus: integral and differential equations of frational order, in A. Carpinteri and F. Mainardi (Editors). *Fractals and Fractional Calculus in Continuum Mechanics*, pages 223 – 276, 1997. Wien and New York, Springher Verlag.

[25] I. S. Gradshteyn and I. M. Ryzhik. *Table of integrals, series and products*. Academic Press, 2007. Seventh edition.

[26] R. Hilfer. Fractional diffusion based on Riemann-Liouville fractional derivatives. *J. Phys. Chem. B*, 104:3914 – 3917, 2000.

[27] H. Hövel and U. Westphal. Fractional powers of closed operators. *Studia Math.*, 42:177 – 194, 1972.

[28] L. F. James. Lamperti type laws. *Ann. App. Probab.*, 20:1303 – 1340, 2010.

[29] A. Kilbas, H. Srivastava, and J. Trujillo. *Theory and applications of fractional differential equations (North-Holland Mathematics Studies)*, volume 204. Elsevier, Amsterdam, 2006.

[30] A. N. Kochubei. The Cauchy problem for evolution equations of fractional order. *Differential Equations*, 25:967 – 974, 1989.

[31] A. N. Kochubei. Diffusion of fractional order. *Lecture Notes in Physics*, 26:485 – 492, 1990.

[32] H. Komatsu. Fractional powers of operators. *Pacific J. Math.*, 19:285 – 346, 1966.

[33] M. A. Krasnosel'skii and P. E. Sobolevskii. Fractional powers of operators acting in Banach spaces. *Doklady Akad. Nauk SSSR*, 129:499 – 502, 1959.

[34] J. Lamperti. An occupation time theorem for a class of stochastic processes. *Trans. Amer. Math. Soc.*, 88:380 – 387, 1963.

[35] N. N. Lebedev. *Special functions and their applications*. Dover, New York, 1972.

[36] F. Mainardi, Y. Luchko, and G. Pagnini. The fundamental solution of the space-time fractional diffusion equation. *Fract. Calc. Appl. Anal.*, 4(2):153 – 192, 2001.

[37] F. Mainardi, G. Pagnini, and R. Gorenflo. Mellin transform and subordination laws in fractional diffusion processes. *Fract. Calc. Appl. Anal.*, 6(4):441 – 459, 2003.

[38] F. Mainardi, G. Pagnini, and R. K. Saxena. Fox H functions in fractional diffusion. *Journal of Computation and Applied Mathematics*, 178:321 – 331, 2005.

[39] F. Mainardi, G. Pagnini, and R. Gorenflo. Some aspects of fractional diffusion equations of single and distributed order. *Applied Mathematics and Computing*, 187:295 – 305, 2007.

[40] F. Mainardi, A. Mura, and G. Pagnini. The M-Wright function in time-fractional diffusion processes: a tutorial survey. *International Juornal of Differential Equations*, 2010. doi:10.1155/2010/104505.

[41] A. Mathai and R. Saxena. *Generalized Hypergeometric functions with applications in statistics and physical sciences.* Lecture Notes in Mathematics, n. 348, 1973.

[42] M. Meerschaert and H. P. Scheffler. Triangular array limits for continuous time random walks. *Stoch. Proc. Appl.*, 118:1606 – 1633, 2008.

[43] M. Meerschaert and H. P. Scheffler. Limit theorems for continuous time random walks with infinite mean waiting times. *J. Appl. Probab.*, 41:623 – 638, 2004.

[44] M. Meerschaert, E. Nane, and P. Vellaisamy. Fractional Cauchy problems on bounded domains. *Ann. Probab.*, 37(3):979 – 1007, 2009.

[45] R. Metzler and J. Klafter. Boundary value problems for fractional diffusion equations. *Physica A*, 278:107 – 125, 2000.

[46] R. Metzler and J. Klafter. The random walk's guide to anomalous diffusion: a fractional dynamics approach. *Phys. Rep.*, 339:1 – 77, 2000.

[47] E. Nane. Fractional Cauchy problems on bounded domains: survey of recent results. *arXiv:1004.1577v1*, 2010.

[48] R. Nigmatullin. The realization of the generalized transfer in a medium with fractal geometry. *Phys. Status Solidi B*, 133:425 – 430, 1986.

[49] E. Orsingher and L. Beghin. Fractional diffusion equations and processes with randomly varying time. *Ann. Probab.*, 37:206 – 249, 2009.

[50] E. Orsingher and X. Zhao. The space-fractional telegraph equation and the related fractional telegraph process. *Chinese Ann. Math. Ser. B*, 24:45 – 56, 2003.

[51] G. Peškir. On the fundamental solution of the Kolmogorov-Shiryaev equation. *From stochastic calculus to mathematical finance*, pages 535 – 546, 2006. Sringher, Berlin.

[52] M. Pollack and D. Siegmund. A diffusion process and its applications to detecting a change in the drift of Brownian motion. *Biometrika*, 72:267 – 280, 1985.

[53] H. Roman and P. Alemany. Continuous-time random walks and the fractional diffusion equation. *J. Phys. A*, 27:3407 – 3410, 1994.

[54] S. Samko. Fractional power of operators via hypersingular integrals. *Progress in Nonlinear Diff. Eq. and Their Applic.*, 42:259 – 272, 2000. Birkhäuser.

[55] S. Samko, A. A. Kilbas, and O. I. Marichev. *Fractional Integrals and Derivatives: Theory and Applications.* Gordon and Breach, Newark, N. J., 1993.

[56] W. Schneider and W. Wyss. Fractional diffusion and wave equations. *J. Math. Phys.*, 30:134 – 144, 1989.

[57] T. Shiga and S. Watanabe. Bessel diffusions as a one-parameter family of diffusion processes. *Z. Wahrscheinlichkeitstheorie Verw. Geb.*, 27:37 – 46, 1973.

[58] V. V. Uchaǐkin. Self-similar anomalous diffusion and Levy-stable laws. *Physics - Uspekhi*, 46(8):821 – 849, 2003.

[59] J. Watanabe. On some properties of fractional powers of linear operators. *Proc. Japan Acad. Ser. A Math. Sci.*, 37:273 – 275, 1961.

[60] W. Wyss. The fractional diffusion equations. *J. Math. Phys.*, 27:2782 – 2785, 1986.

[61] G. Zaslavsky. Fractional kinetic equation for Hamiltonian chaos. *Phys. D*, 76: 110 – 122, 1994.

[62] V. M. Zolotarev. Mellin-Stieltjes transformations in probability theory. *Teor. Veroyatnost. i Primenen*, 2:444 – 469, 1957. Russian.

[63] V. M. Zolotarev. *One-dimensional stable distributions, volume 65 of Translations of Mathematical Monographs.* American Mathematical Society, 1986. ISBN 0-8218-4519-5. Translated from the Russian by H. H. McFaden, Translation edited by Ben Silver.